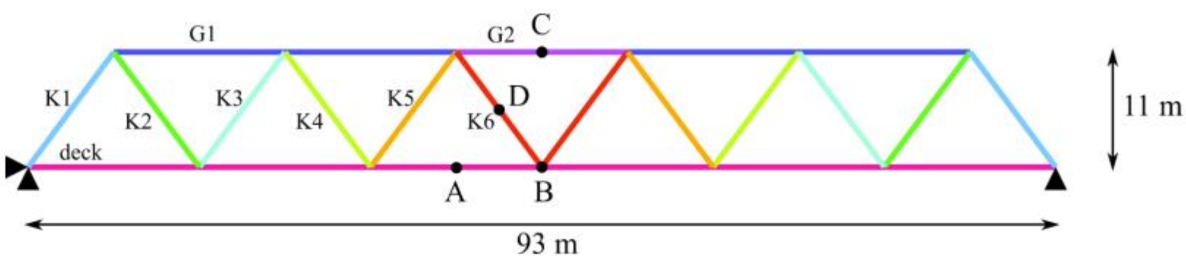
Assignment2

In-plane Finite Element Model of a Truss Bridge

Objective

Using a finite element model (FEM) to model the truss bridge. As shown in the figure, we used FEM to calculate the natural frequencies, vibration modes, damping matrix, and frequency response function (FRF) for an input force at point A. FRF of the structure after TMD implementation, structural static response, and critical speed of a long passing train.



Given data:

	Colour	m [kg/m]	EA	EJ
Deck Element	Pink	2.36E+03	1.59E+10	1.36E+09
Diag member K1	Light blue	4.42E+02	1.18E+10	2.80E+08
Diag member K2	Green	2.56E+02	6.85E+09	1.36E+08
Diag member K3	Cyan	2.75E+02	7.35E+09	1.51E+08
Diag member K4	Light green	1.75E+02	4.68E+09	4.48E+07
Diag member K5	Orange	1.45E+02	3.89E+09	6.13E+07
Diag member K6	Red	94.2	2.52E+09	1.63E+07
Upper chord G1	Blue	5.34E+02	1.43E+10	1.21E+09
Upper chord G2	Purple	5.93E+02	1.59E+10	1.36E+09

The procedure for developing the FE model is as follows:

1) Mesh generation

2) Definition of the global and local reference systems

3) Removal of external constraints and introduction of corresponding constraint forces

4) Energy functions formulation in the local nodal coordinates of each element

5) Coordinate transformation from the local to the global reference system

6) Matrix assembling, for the entire structure

7) Model of the structural damping

8) Re-introduction of external constraints and matrix partition

1. Define a FE model of the structure in the 0 – 7 Hz frequency range (check that for each element k, $\omega 1,k$ $\Omega max \geq 1.5$).

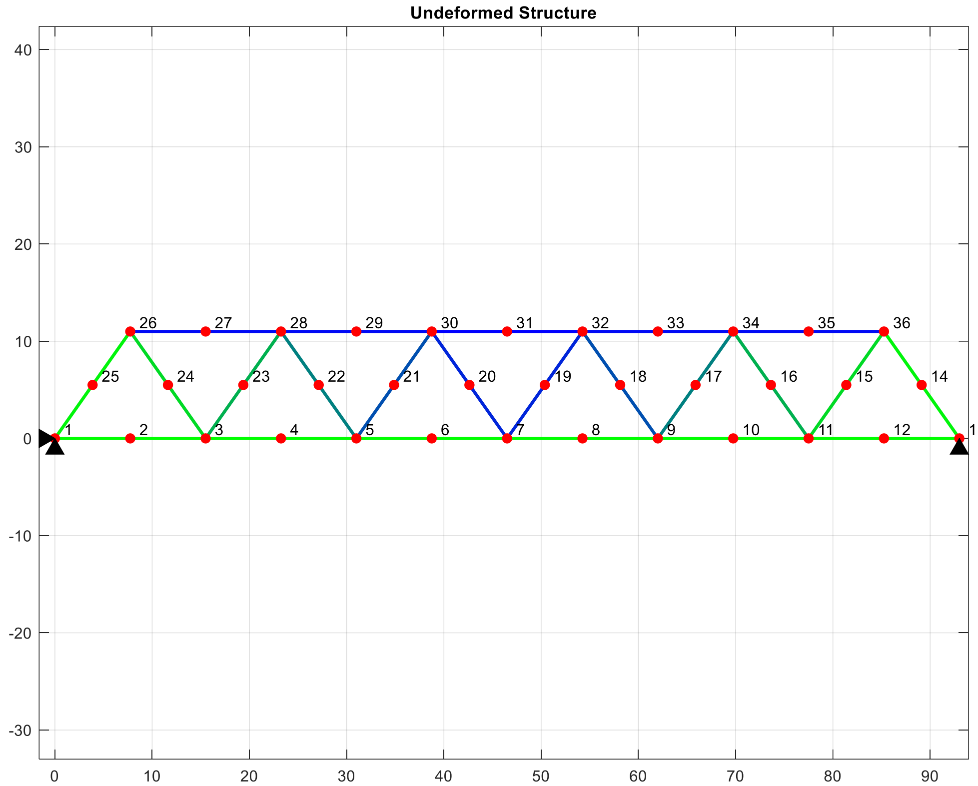
First, the model of the bridge is to be checked that every element of the structure is working in the quasi-static region for the frequency range of interest which means the first natural frequency of each beam element ω_k^1 must be higher than the maximum frequency at which the system can be excited.

$$\omega_k^{(1)} = (\frac{\pi}{L_k})^2\sqrt{\frac{EJ}{m}} \qquad \text{Where,} \qquad \frac{\omega_k^{(1)}}{\Omega_{max}} \geq 1.5$$

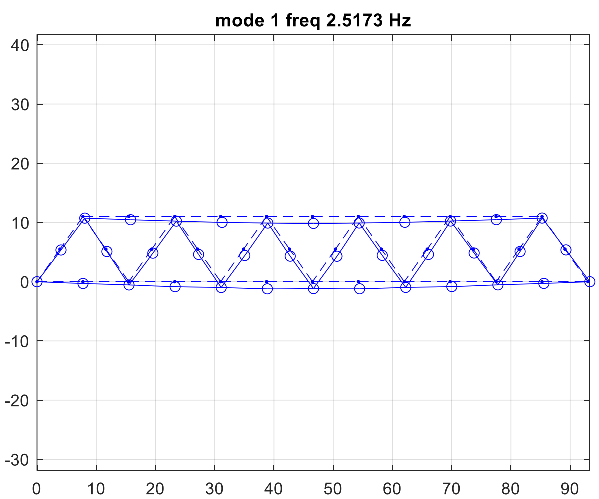
The maximum length for each element is achieved by the below formula.

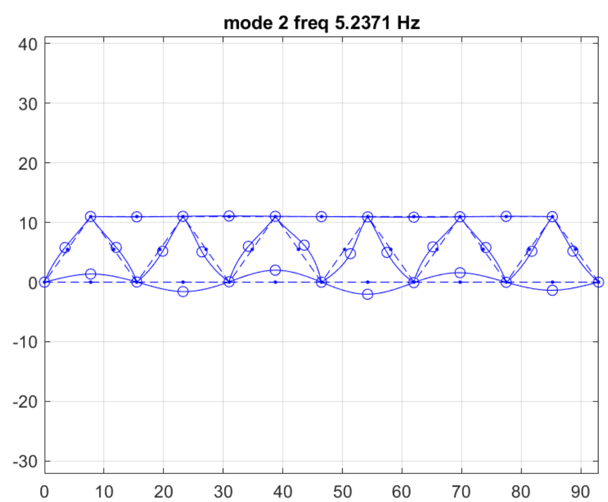
$$L_{max} = \sqrt{\frac{\pi^2}{1.5\ \Omega_{max}}\sqrt{\left(\frac{EJ}{m}\right)_{min}}}$$

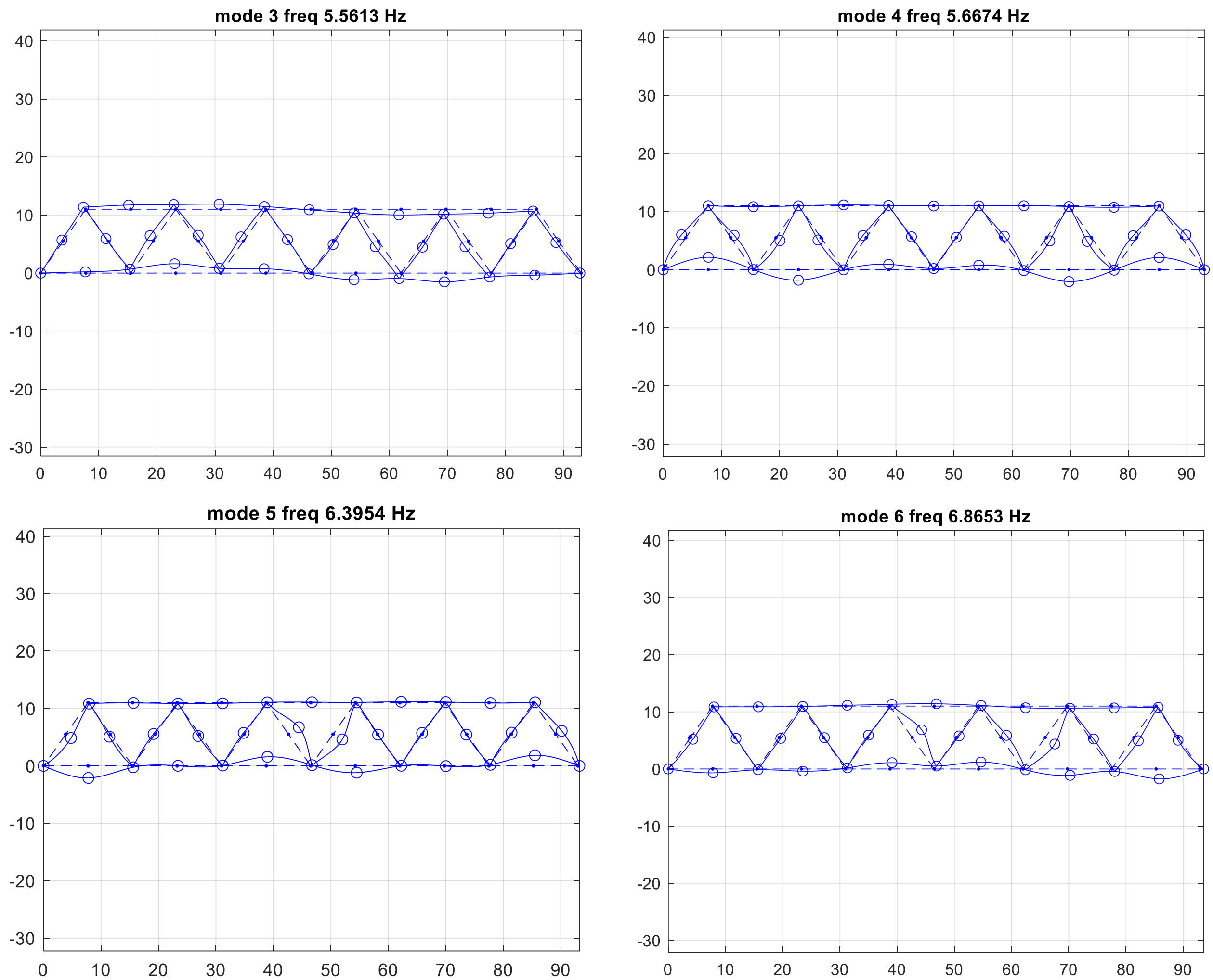
Considering the limitation of length we impose nodes in the middles, ends and, intersections, we obtain 36 nodes located at each joint and middle of each truss as in the figure below:



2. Calculate the structure’s natural frequencies and vibration modes up to the 6th mode. Plot the mode shapes with the indication of the associated natural frequencies.







3. Compute the damping matrix of the system according to the proportional damping assumption: [C] = α[M] + β[K], where α and β have to be fixed such that they result in the following damping ratios for the first two vibration modes: ξ1 = 1%, ξ2 = 0.75%

Considering damping ratios of $\xi_1 = 1\%$ and $\xi_2 = 0.75\%$ it is now possible to compute the damping matrix [C] of the system by assuming it to be written as a linear combination of the mass [M] and stiffness [K] matrices.

$$[\mathbf{C}] = \alpha[\mathbf{M}] + \beta[\mathbf{K}]$$

For two variables, we required two equations, hence we consider two modes of the system to compute the values of **α** and **β**.

$$c_1 = \alpha m_1 + \beta k_1$$

$$c_2 = \alpha m_2 + \beta k_2 \,;$$

$$\xi_1 = \frac{\alpha}{2\omega_1} + \frac{\beta\omega_1}{2}$$
$$\xi_2 = \frac{\alpha}{2\omega_2} + \frac{\beta\omega_2}{2}$$

Since, $\xi_i = \dfrac{c_i}{2m_i\omega_i}$; $\omega_i = \sqrt{\dfrac{k_i}{m_i}}$

The linear system can be written in matrix form as :

$$\begin{bmatrix} \frac{1}{2\omega_1} & \frac{\omega_1}{2} \\ \frac{1}{2\omega_2} & \frac{\omega_1}{2} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Solving for $\boldsymbol{\alpha}$ **,** $\boldsymbol{\beta}$ with $\xi_1 = 1\%$ and $\xi_2 = 0.75\%$,

$$\boldsymbol{\alpha \approx 0.263116}$$

$$\boldsymbol{\beta \approx 2.13 \cdot 10^{-4}}$$

4. Calculate the frequency response functions which relate the input force (vertical force in node A) to the following outputs.

a) Vertical displacement and vertical acceleration of nodes A, B and C. Plot the corresponding magnitude and phase diagrams.

b) Shear force, bending moment and axial force evaluated in nodes C and D. Plot the corresponding magnitude and phase diagrams.

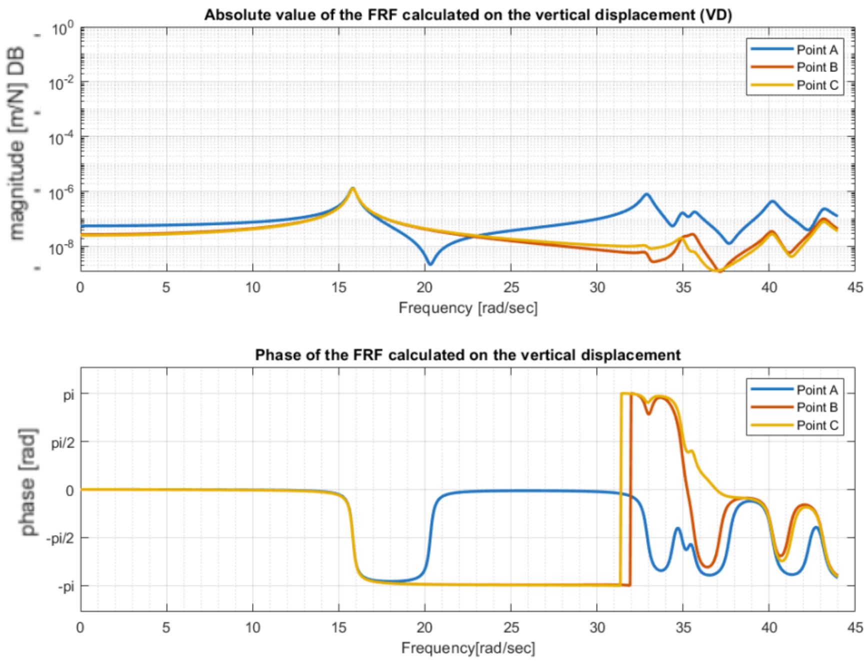
c) Constraint forces. Plot the corresponding magnitude and phase diagrams.

Assume the input force to vary in the 0 – 7 Hz frequency range and set the frequency resolution to 0.01 Hz.

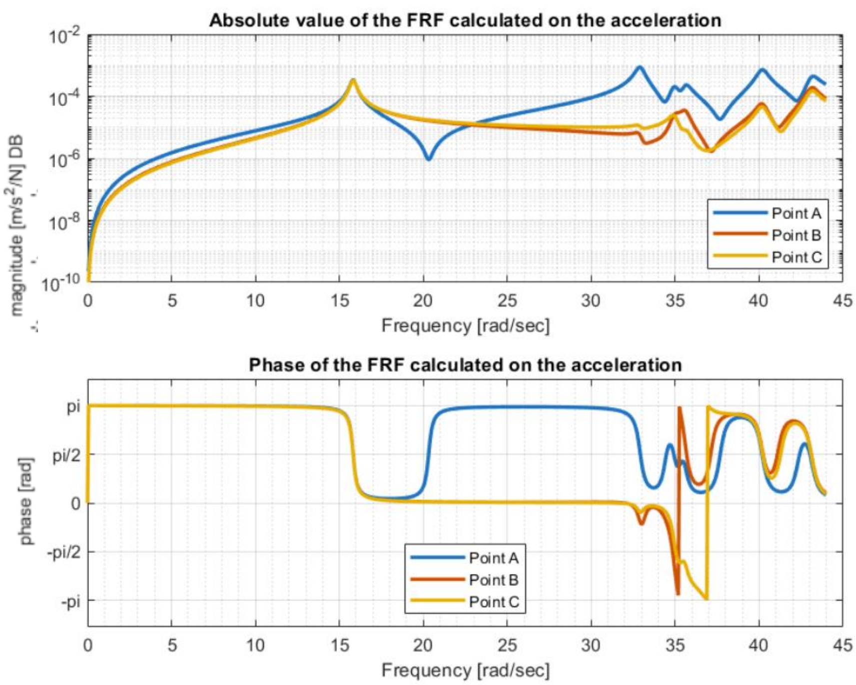
a) Vertical displacement and vertical acceleration of nodes A, B and C. Plot the corresponding magnitude and phase diagrams.

For the computation of the FRFs, a vertical harmonic input force $F = F0e\ i\Omega t$ applied in node A is considered with outputs calculated at A, B and C. Since the force is applied at a nodal point A we won't need to rely on shape functions to obtain the equivalent nodal force. The FRF can be computed as

$$(-\Omega^2[M_{FF}] + j\Omega[C_{FF}] + [K_{FF}])\underline{X}e^{j\Omega t} = \underline{F_0}e^{j\Omega t}$$



Vertical displacement



Vertical Acceleration

b) Shear force, bending moment and axial force evaluated in nodes C and D. Plot the corresponding magnitude and phase diagrams.

The nodal displacements through the shape functions are used to define the axial and transverse displacement of a beam section.

Axial displacement : $u(x, t) = a + bx$ - Linear function

Transverse displacement: $w(x, t) = a + bx + cx^2 + dx^4$ - Cubic function

The coefficients a, b, c, and d were found by imposing the boundary conditions on the shape functions.

In an Euler-Bernoulli beam:

$$N = EA\frac{\partial u}{\partial \xi}$$

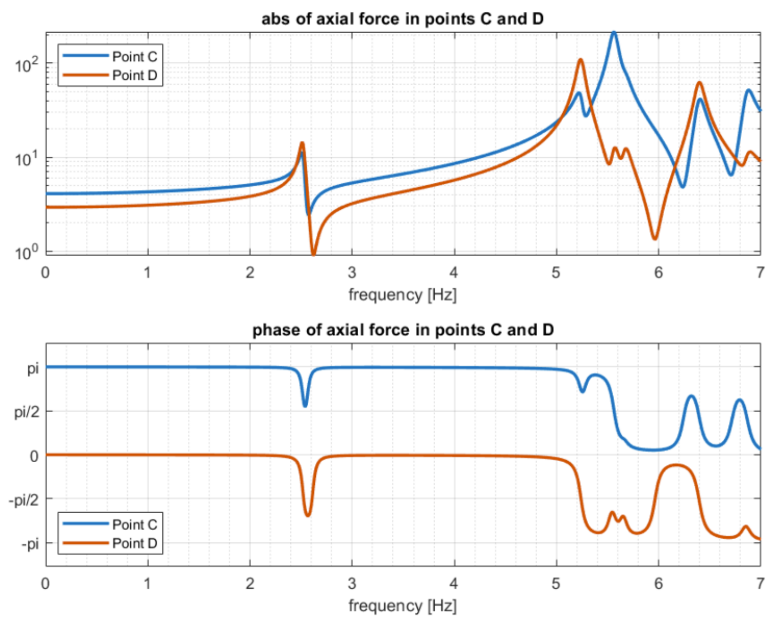
$$M = EJ\,\frac{\partial^2 w}{\partial \xi^2}$$

$$T = EJ\,\frac{\delta^3 w}{\delta \xi^3}$$

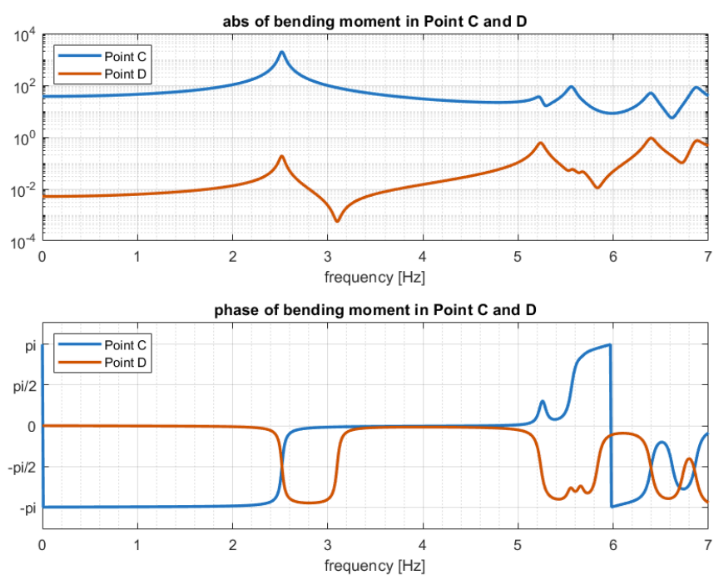
Where,

ξ is local coordinate inside the beam element.

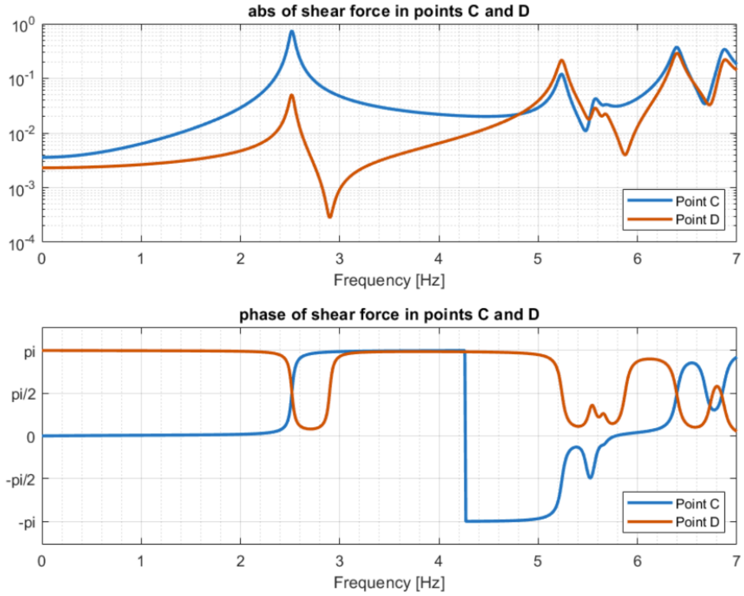
Knowing how to compute these parameters it is possible to plot the FRFs functions.



Axial Force



Bending Moment



Shear Force

c) Constraint forces. Plot the corresponding magnitude and phase diagrams.

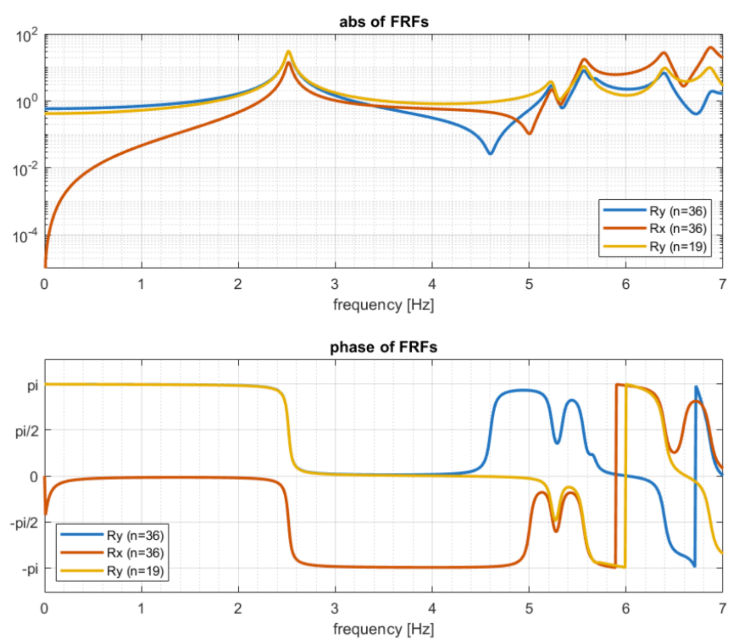
When the response is known, the constraint forces can be computed as:

$$[M_{FC}]\underline{\ddot{x}_F} + [C_{FC}]\underline{\dot{x}_F} + [K_{FC}]\underline{x_F} = \underline{R} \qquad \underline{x_F} = \underline{x_0}e^{j\Omega t} = \underline{G_{disp}}F_0 e^{j\Omega t};$$

$$\underline{R} = \underline{R_0}e^{j\Omega t}$$

$$(-\Omega^2[M_{FC}] + j\Omega[C_{FC}] + [K_{FC}])\frac{\underline{x_0}}{F_0} = \frac{\underline{R}}{F_0}$$

$$\text{FRF} \quad \rightarrow \quad (-\Omega^2[M_{FC}] + j\Omega[C_{FC}] + [K_{FC}])\underline{G_{disp}}(j\Omega) = \frac{\underline{R}}{F_0}$$



5. Using the modal superposition approach and considering the structure's first three modes, calculate the frequency response functions which relate the same input force of question 4 (vertical force applied in node A) to the vertical displacement and vertical acceleration of nodes A, B and C. Plot the corresponding magnitude and phase diagrams superimposed to those obtained in item 4a. Point out the differences and comment on the results.

Using the modal superposition method, and considering the structure's first three modes:

$$[\phi] = \begin{bmatrix} \underline{X}^{(1)} & \underline{X}^{(2)} & \underline{X}^{(3)} \end{bmatrix}$$

$$[M_q] = [\phi]^T [M][\phi]$$

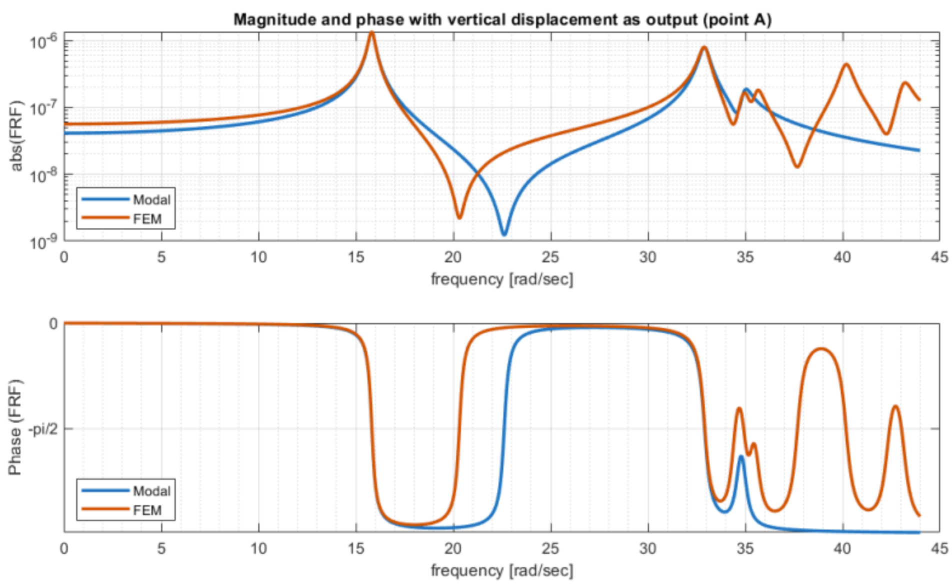
$$[K_q] = [\phi]^T [K][\phi]$$

$$[C_q] = [\phi]^T [C][\phi]$$

$$[Q_q] = [\phi]^T \underline{F}$$

$$[M_{FF}]\underline{\ddot{x}}_F + [C_{FF}]\underline{\dot{x}}_F + [K_{FF}]\underline{x}_F = \underline{F}_F^G$$

The FRFs are computed for each output node location A, B and C

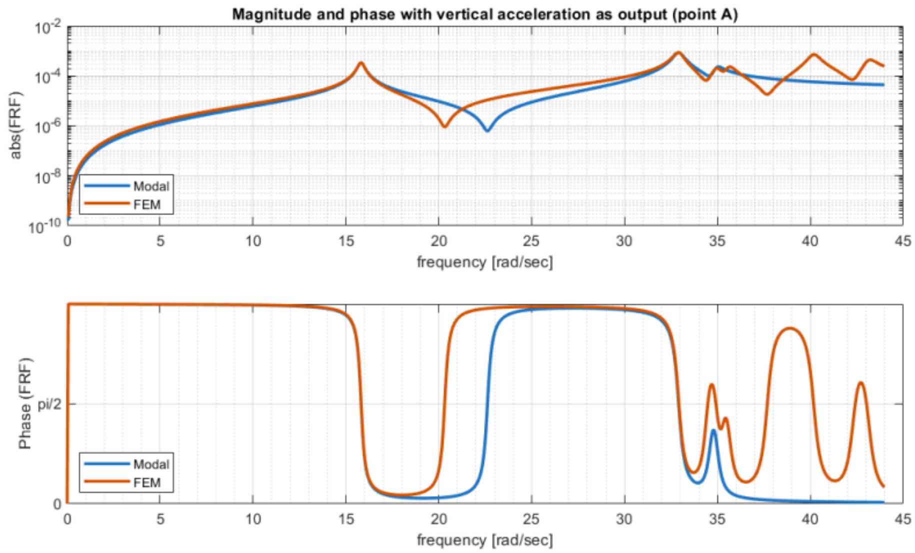


Point A Vertical displacement.

As can be seen, there is a small difference between the two methods for low frequencies which is explained by the presence of high-frequency contributions that are still present. Since the identified natural frequencies (peaks) are the same in both methods, it can be concluded that the identified natural frequencies are accurate.

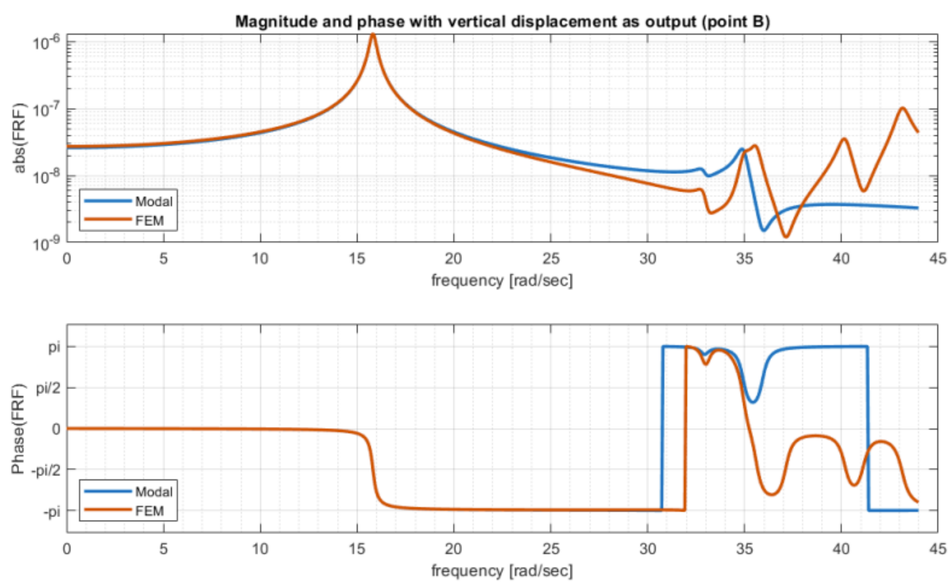
Additionally, there are significant differences between the two approaches for high frequencies. Since only the first three modes are taken into account in the model superposition approach, we can expect that the obtained FRF will be unreliable at non-resonating frequencies. Hence results in FEM are more reliable for higher frequencies.

Similar conclusions can be derived for the FRF plots of points B and C.

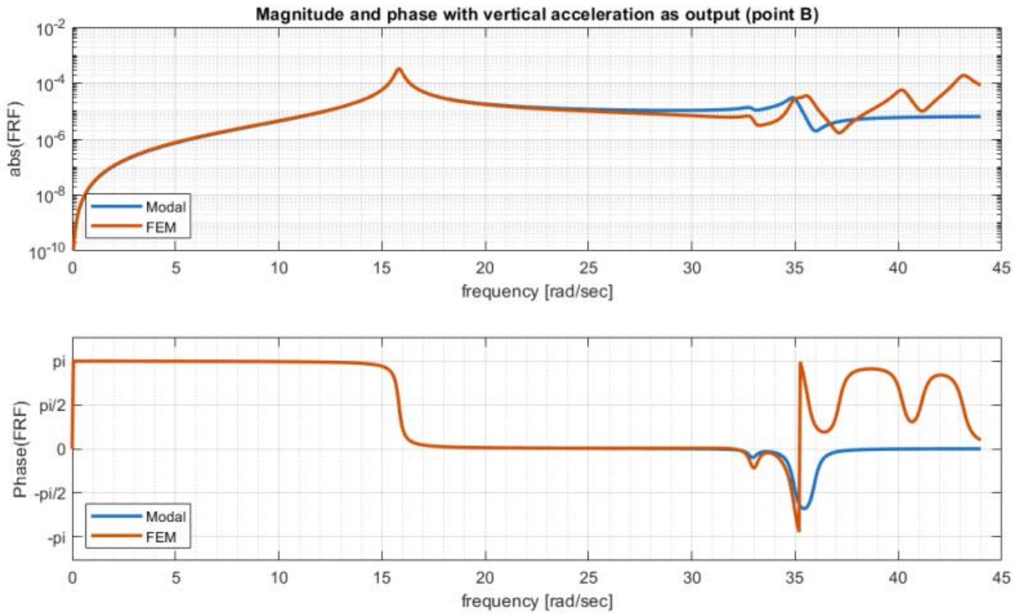


Point A Vertical Acceleration

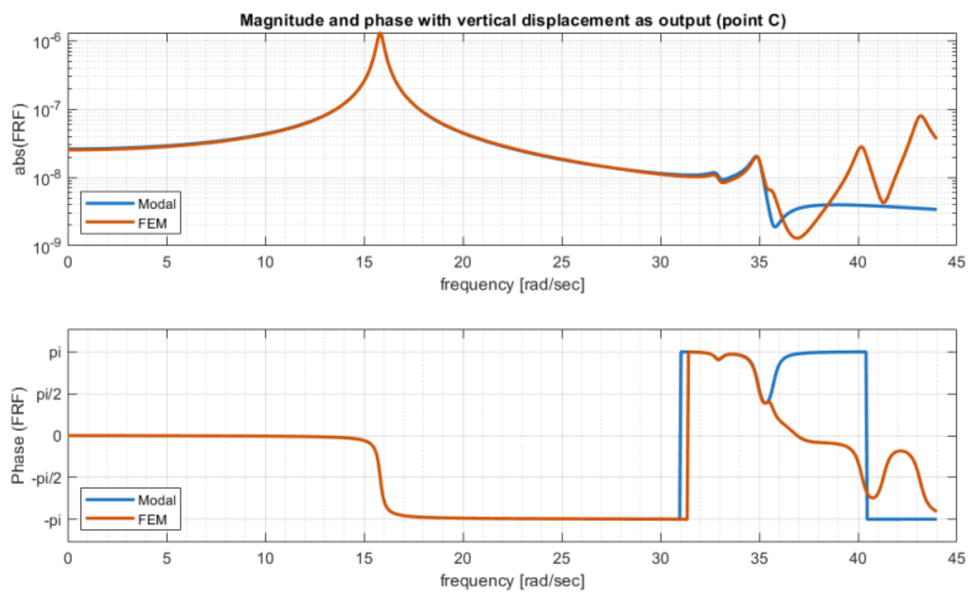
If we look at the evaluation with acceleration as the output, we can say that both methods are highly accurate for low frequencies, but they disagree for high frequencies.



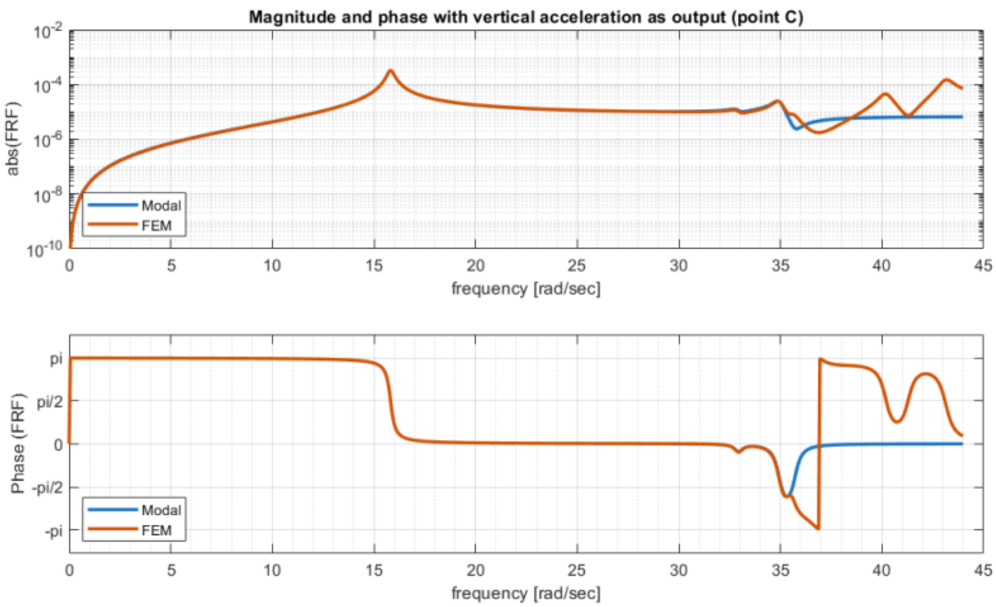
Point B Vertical Displacement



Point B Vertical Acceleration



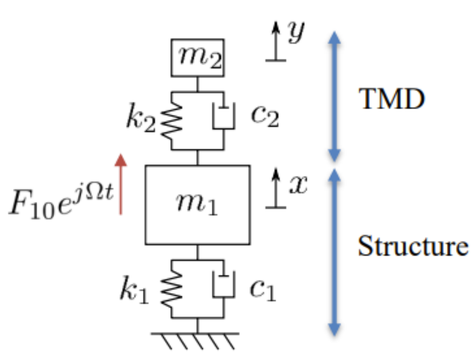
Point C Vertical Displacement



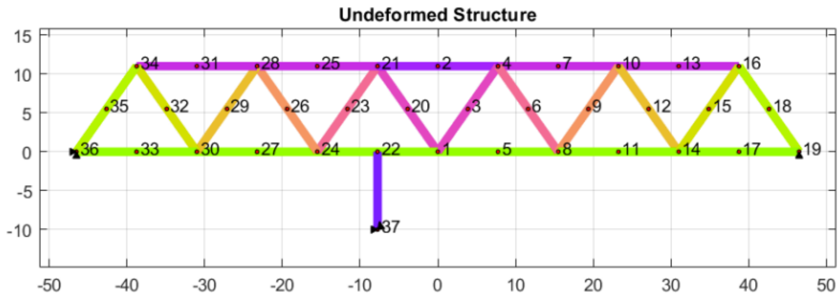
Point C Vertical Acceleration

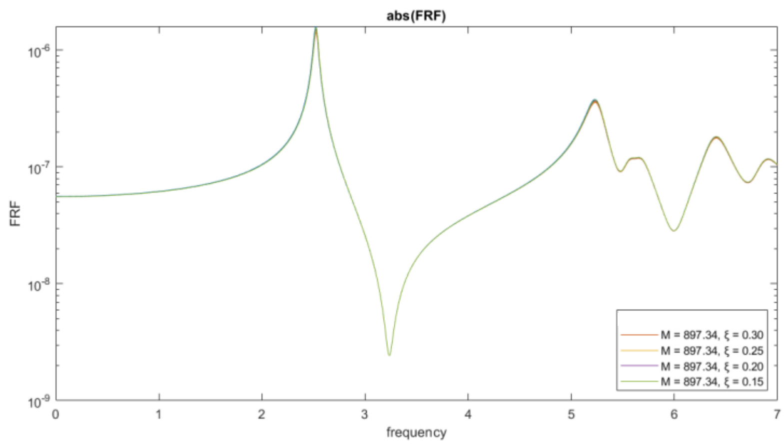
6. Design a TMD to be installed on the structure so as to reduce by at least 15% the amplitude of the vertical vibration of node A in correspondence with the resonance peak of the first mode of vibration, for the same excitation condition of question 4. The increase of the structure mass must be limited to a maximum of 2% while the damping ratio of the TMD alone must not exceed 30%. Provide a plot comparing the FRF results corresponding to the structure with and without the TMD.

A tuned mass damper (TMD) is a mechanical device used in reducing vibrations and improving the performance of structures under dynamic loads. It is designed to counteract the resonant vibrations that can occur due to external forces such as wind, earthquakes, or mechanical disturbances.



New structure with TDM mounted on node 22.

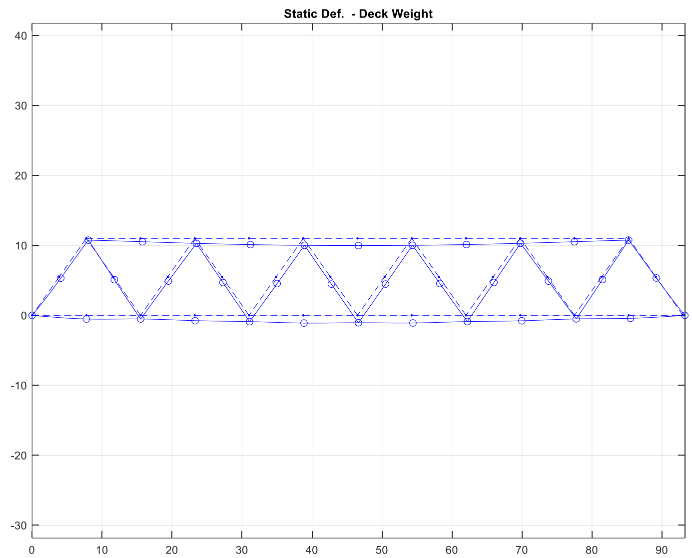




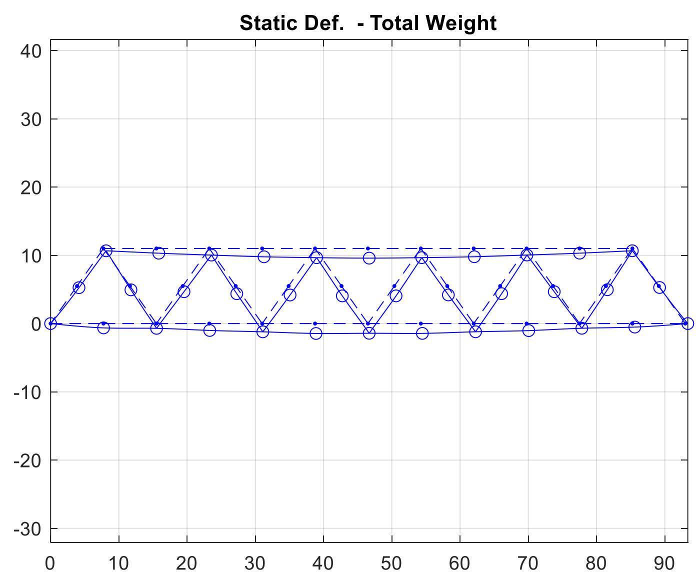
	Max FRF with TMD	**Amplitude Reduction**
m = 897,34 kg and ξ = 0,30	1.41E-06	21.47%
m = 747,78 kg and ξ = 0,30	1.48E-06	17.99%
m = 598,22 kg and ξ = 0,30	1.54E-06	14.40%
m = 448,67 kg and ξ = 0,30	1.61E-06	10.74%
m = 897,34 kg and ξ = 0,30	1.41E-06	21.47%
m = 897,34 kg and ξ = 0,25	1.46E-06	18.97%
m = 897,34 kg and ξ = 0,20	1.51E-06	16.31%
m = 897,34 kg and ξ = 0,15	1.56E-06	13.47%

7. Compute the static response of the structure due to: a) the deck weight only; b) the weight of the entire structure. Plot the deformed shape of the structure compared to the undeformed configuration and compute the value of the maximum deflection.

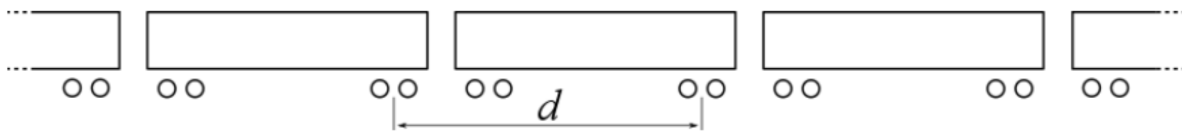
Maximum displacement = **0.9mm**



Maximum displacement = **1.2 mm**



8. Assuming that the bridge is being crossed by a long train, compute the critical velocities for the first two modes considering a value for the distance d of 27.5 m



For the above problem, the critical velocities for the first two modes need to be determined.

The load applied by the train can be compared to a series of moving loads that are evenly spaced apart by d = 27.5m, and the generalized force $Q_i(t)$ can be calculated by adding the contributions of each load in the series.

$$Q_i(t) = \sum_{k=0}^{\infty} Q_{ik} cos(k\Omega_0 t + \Psi_{ik})$$

Where, $$\Omega_0 = \frac{2\pi V}{d}$$

Which gives critical velocity at resonance as, $V_{ik} = \frac{f_i d}{k}$

Where V_{ik} is the critical velocity of the structure whenever the i^{th} mode is excited in resonance when the circular frequency Ω_0 of the k^{th} harmonic component of the generalized force $Q_i(t)$ equals the natural frequency i of the beam.

The critical velocity for the first two natural frequencies are

	V_{i1}	V_{i2}	V_{i3}
V_{1k}	57	28	19
V_{2k}	118	69	39