To begin with, the linearized equation of the mechanical system motion is obtained in the exercise session which is:

$$\left(m\frac{L^2}{4}+J\right)\ddot{\theta}+\left(c\frac{L^2}{4}\right)\dot{\theta}+\left(K\frac{L^2}{4}-mg\frac{L}{2}\right)\theta=C$$

$$m^*=\left(m\frac{L^2}{4}+J\right) \quad ; \quad c^*=\left(c\frac{L^2}{4}\right) \quad ; \quad k^*=\left(K\frac{L^2}{4}-mg\frac{L}{2}\right)$$

$$\rightarrow\ m^*\ddot{\theta}+c^*\dot{\theta}+k^*\theta=C$$

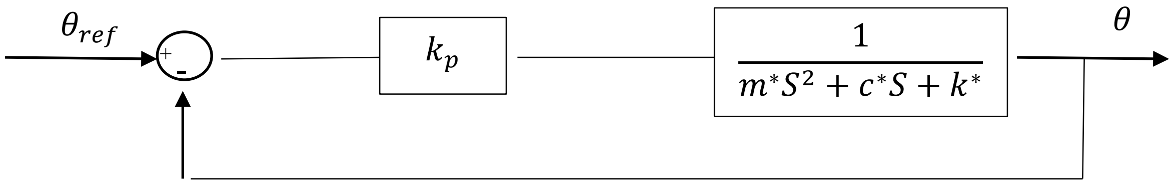
1) Proportional Controller:

The equation for this type of controller acting on torque C is $C=k_p(\theta_{ref}-\theta)$.

Equation and block diagram in Laplace domain are defined in this way:

$$(m^*S^2+c^*S+k^*)\,\theta(S)=C(S)$$

$$C(S)=k_p(\theta_{ref}(S)-\theta(S))$$



The open loop transfer function for analyzing stability is needed which is:

$$G(S)=\frac{k_p}{(m^*S^2+c^*S+k^*)}$$

For studying stability in time domain, we know that m^* and c^* are always positive. Hence, the important factor for recognition the <u>Closed loop system</u> as stable is k^*.

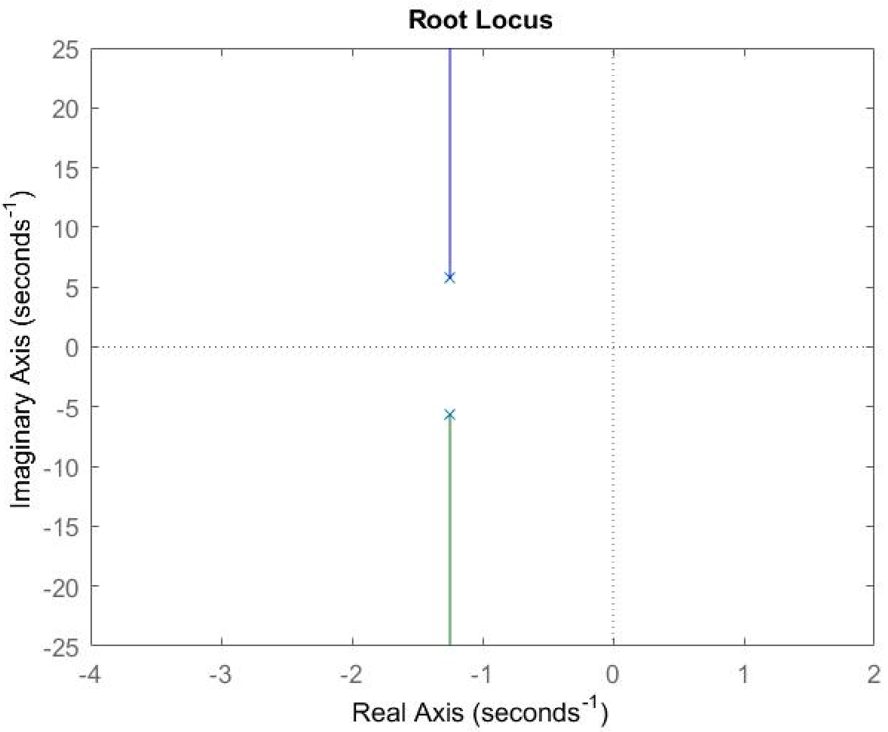
If $k^*>0\ \rightarrow$
$system\ will\ be\ stable\ (two\ complex\ poles(eigen\ values)\ with\ positive\ reals)$

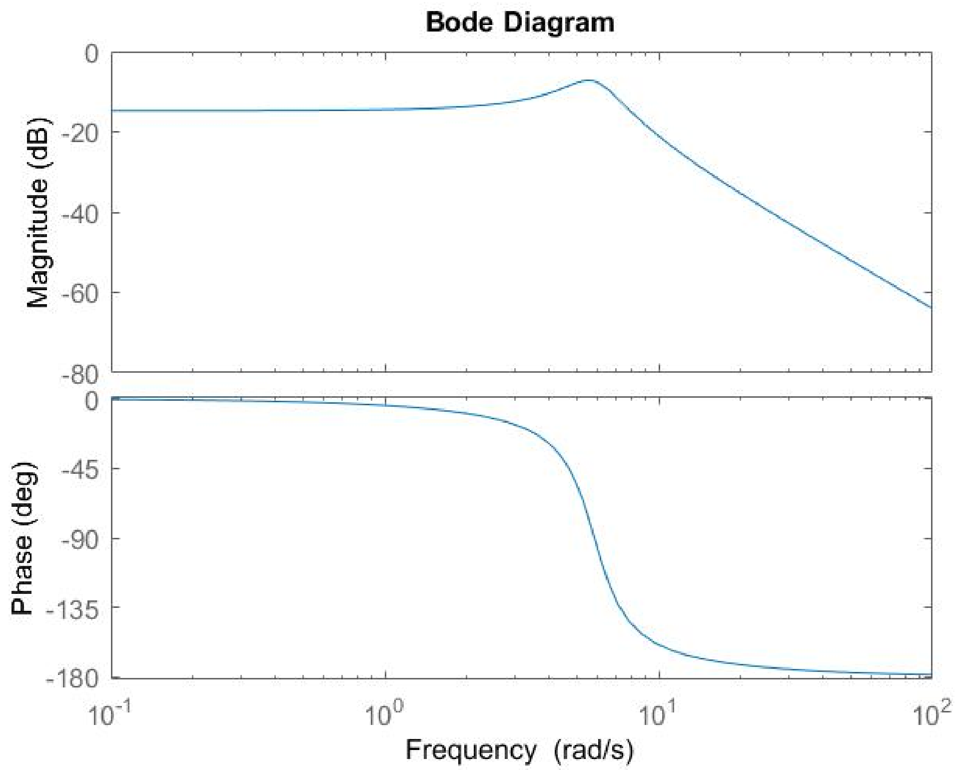
If $k^*=0\ \rightarrow$
$system\ will\ be\ stable(\ one\ pole\ at\ origin\ and\ another\ negative\ real\ pole)$

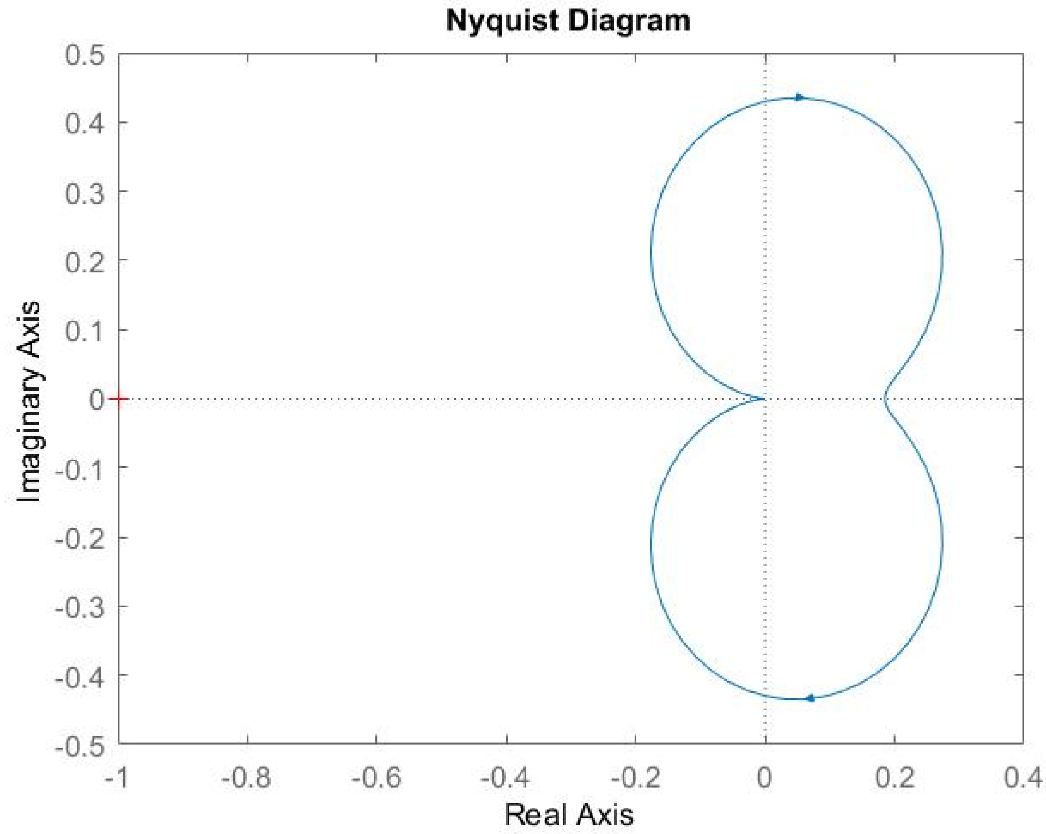
If $k^*<0\ \rightarrow$
$system\ will\ be\ unstable\ (one\ positive\ real\ and\ one\ negative\ real\ pole)$

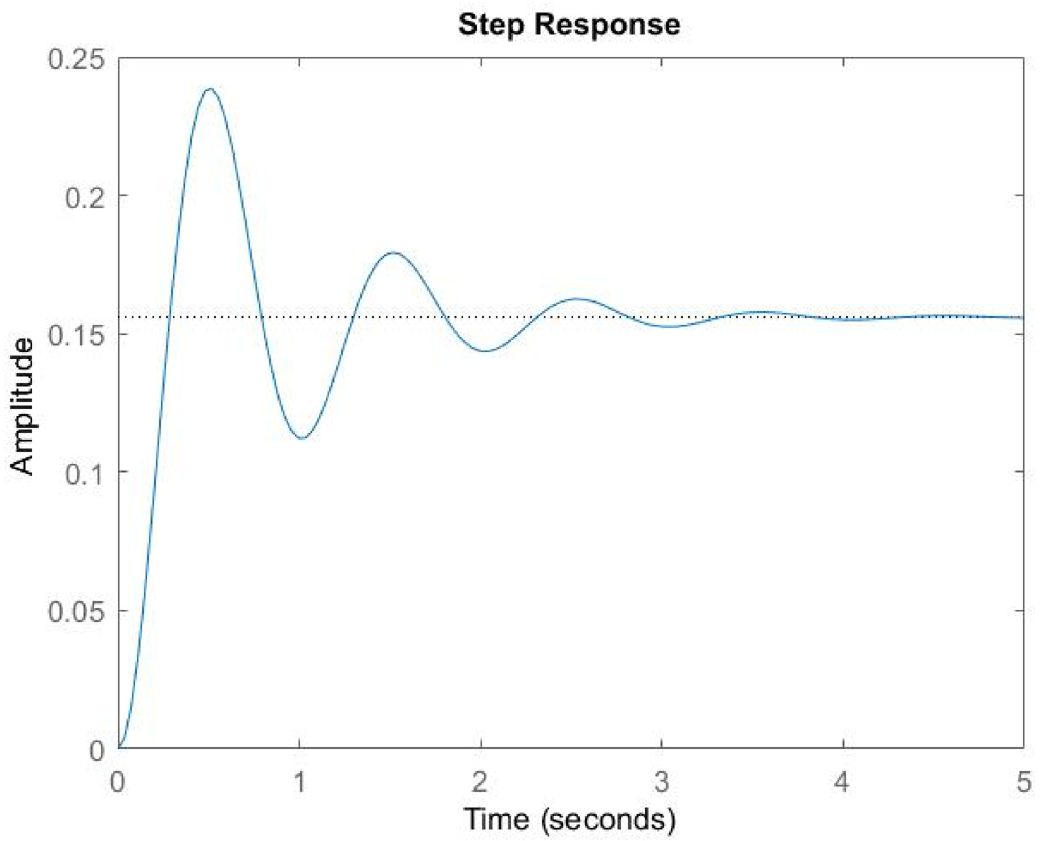
$$\begin{cases} k_1=150\ \rightarrow\ k^*>0 \\ k_2=\dfrac{mg\frac{L}{2}}{K\frac{L^2}{4}}=14.71\rightarrow\ k^*=0 \\ k_3=13\ \rightarrow\ k^*<0 \end{cases}$$

❖ Case A) We consider k_1 as the stiffness of the system. Here are the outcomes and responses of the system both in time and Laplace domain resulted by Matlab (we assume $k_p = 100$, and then see the effect of increasing gain):






Nyquist Diagram
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Imaginary Axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
Real Axis

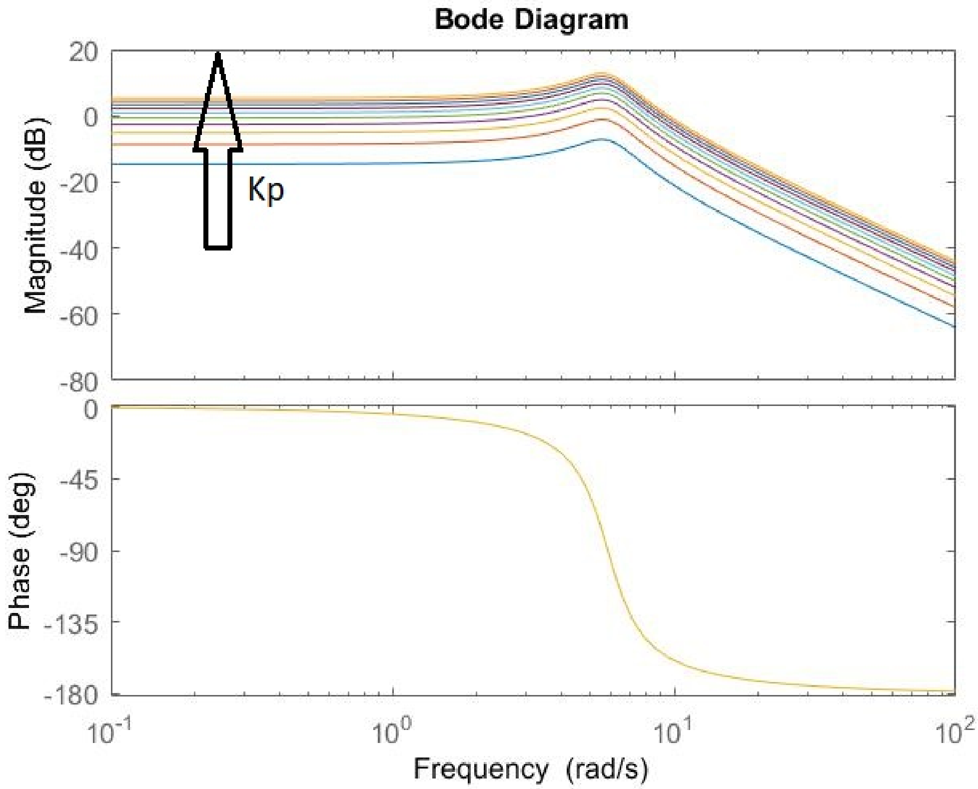

Step Response
0.25
0.2
0.15
0.1
0.05
0
Amplitude
0
1
2
3
4
5
Time (seconds)

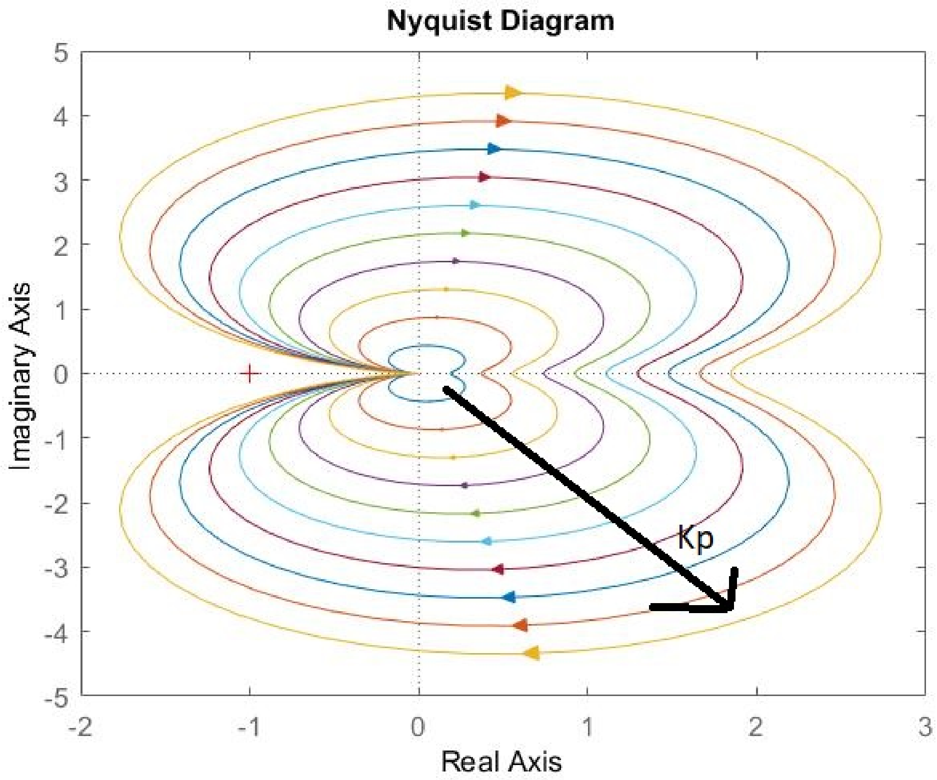
As expected, system is stable for any value of k_p, and stability criterion is satisfied in all of the approaches (bode, nyquist, root locus).

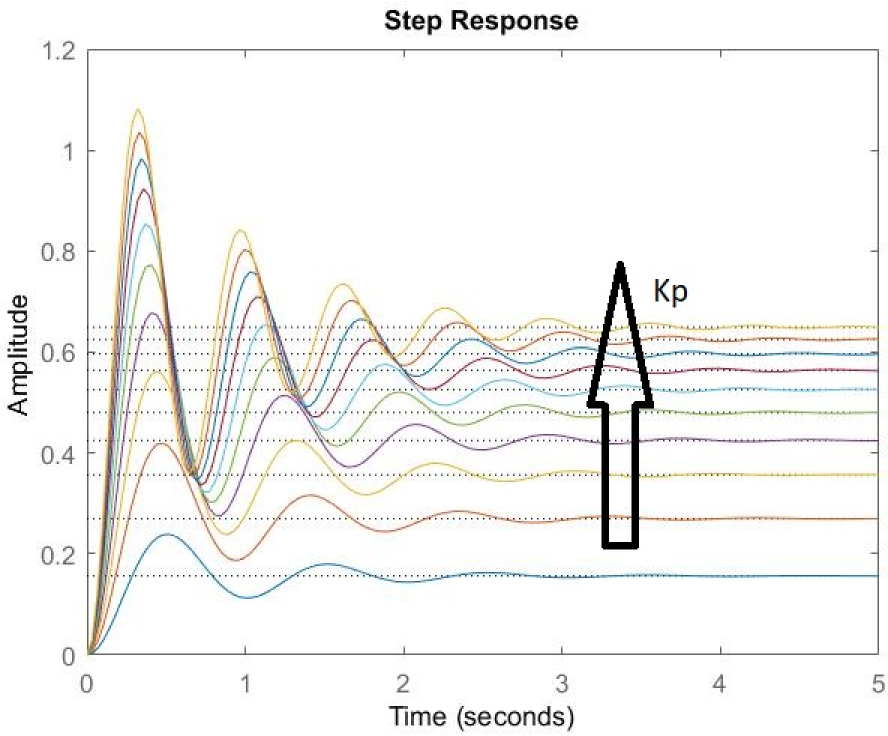
$$\begin{cases} bode: PM > 0 \ , GM \ \rightarrow +\infty \\ nyquist: P = 0 \, , N = 0 \ \rightarrow P = N \end{cases}$$

However, from the last plot it is obvious that system cannot follow the step input signal by having only P controller. Therefore, steady-state error in this condition is high. ($e_{ss} = \theta_{ref} - \theta_{\infty} \cong 1 - 0.156 = 0.844$)

<u>Effect of increasing values of gain:</u> Increasing the gain of the controller from 100 to 1000, and seeing the results.

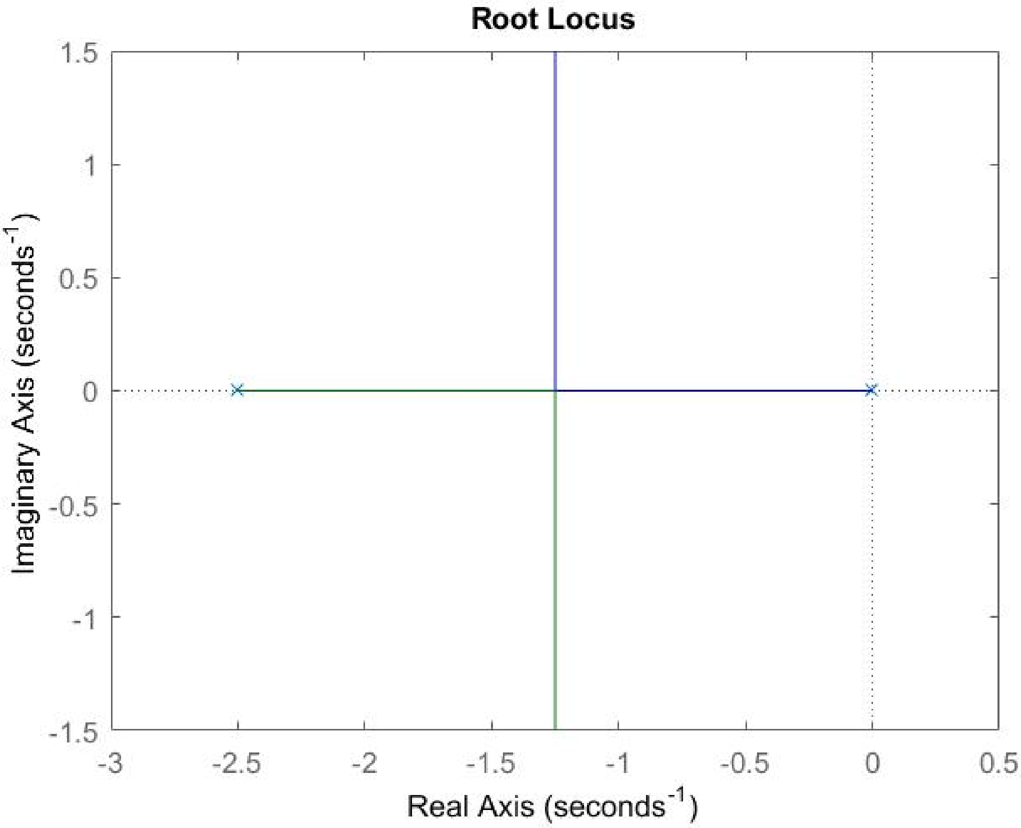


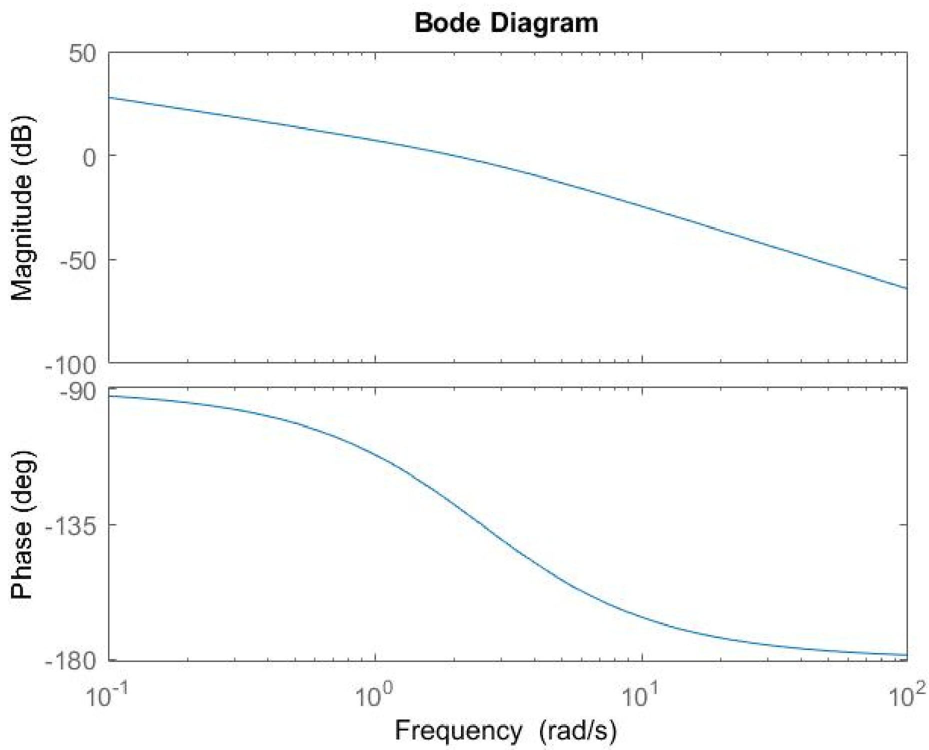


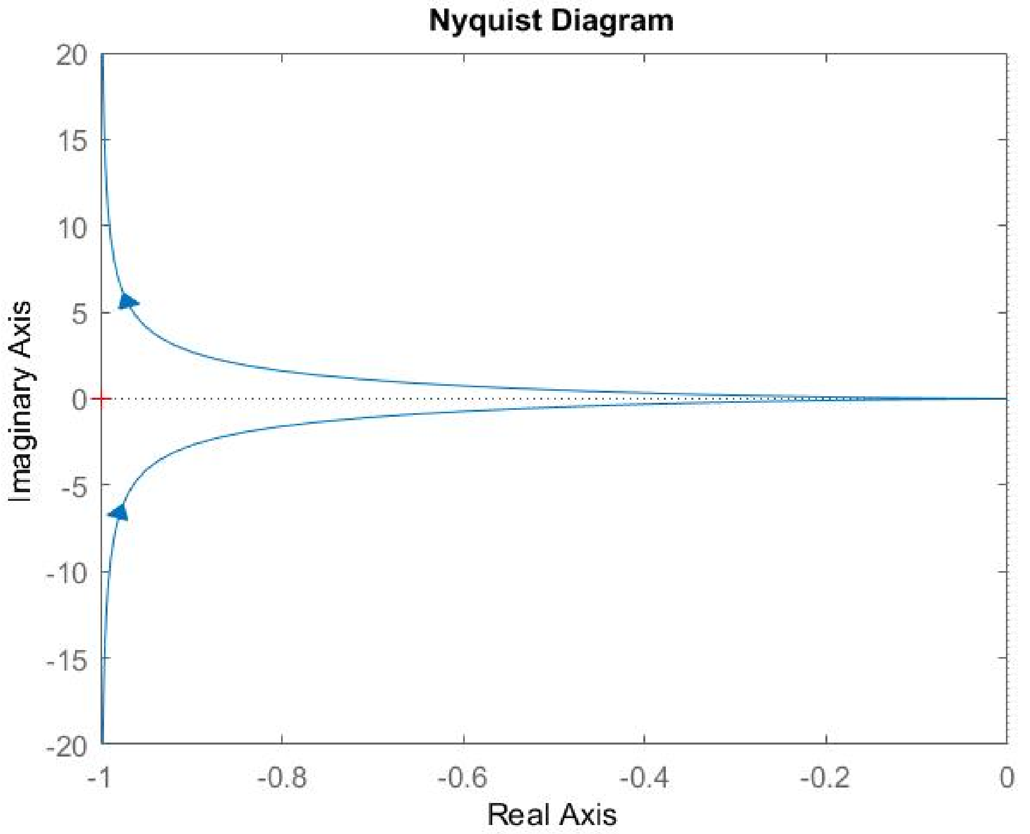


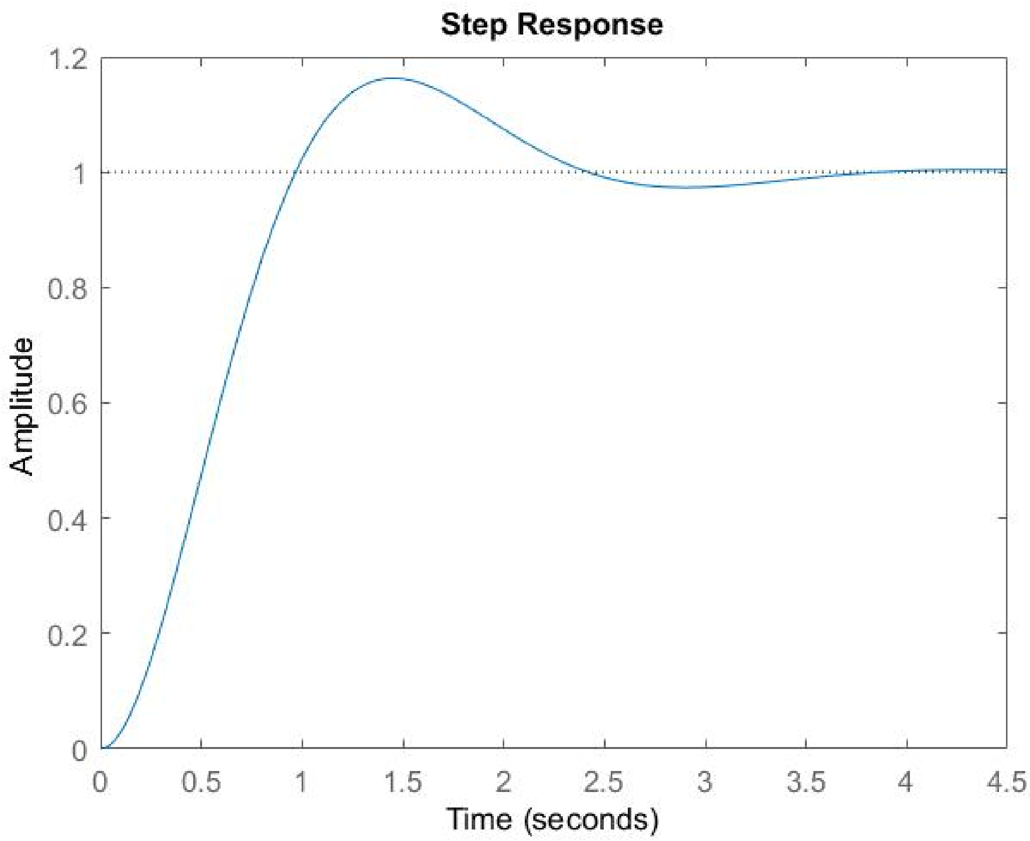
Theoretically, we expect increase of overshoot, natural and damped frequency, and decrease of rise time and steady-state error by enhancing of k_p value. So, we can see these changes in above plots which happens by increasing the gain. (These comments are also valid for Case B&C)

❖ Case B) We consider k_2 as the stiffness of the system. Here are the outcomes and responses of the system both in time and Laplace domain resulted by Matlab (we assume $k_p = 100$, and then see the effect of increasing gain):






Nyquist Diagram
20
15
10
5
0
-5
-10
-15
-20
Imaginary Axis
-1
-0.8
-0.6
-0.4
-0.2
0
Real Axis

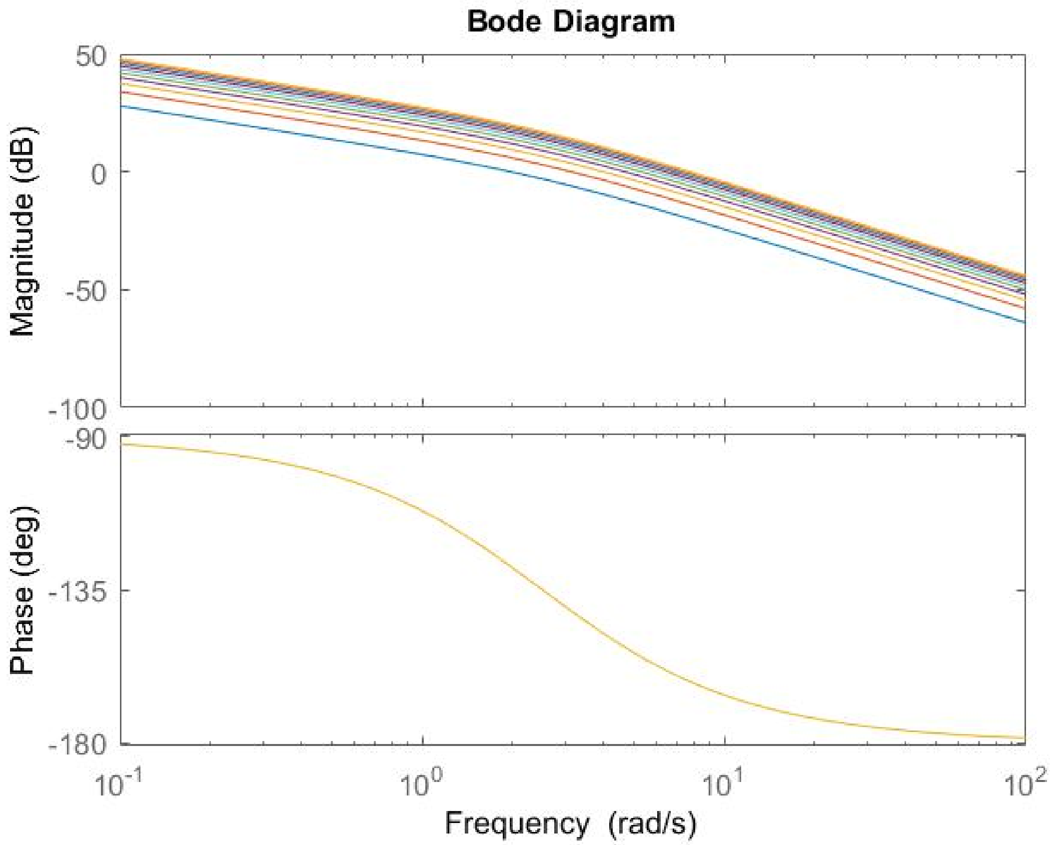

Step Response
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
Time (seconds)

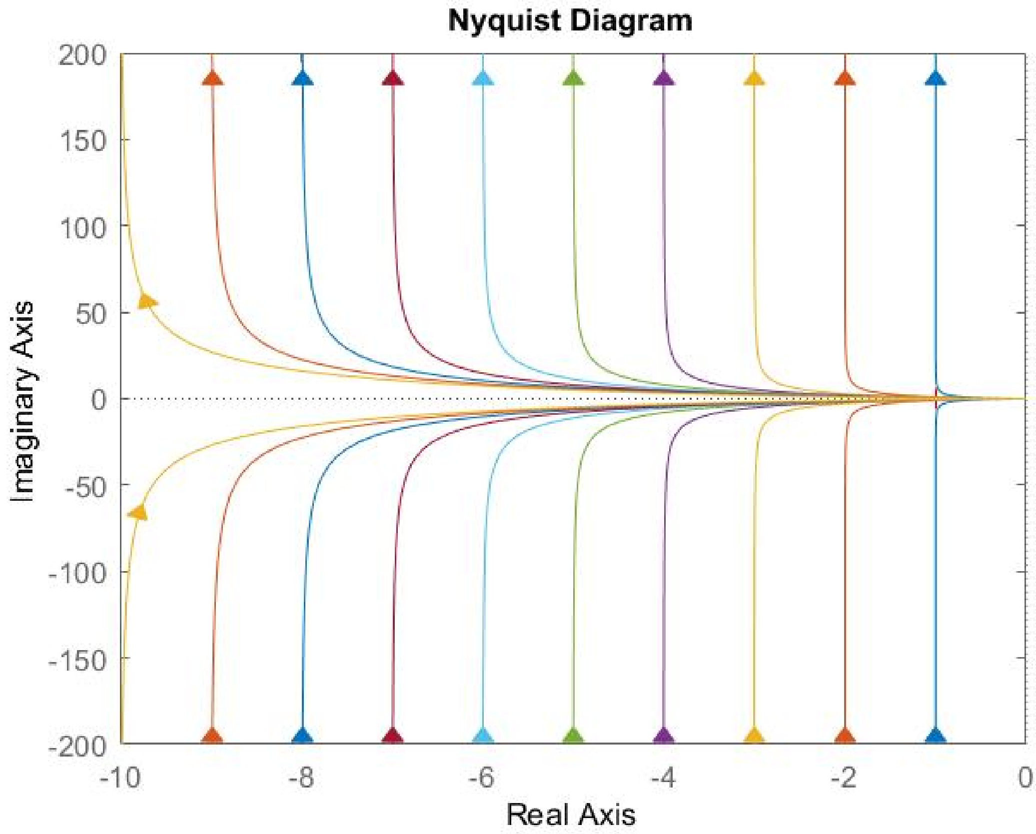
As expected, system is stable for any value of k_p, and stability criterion is satisfied in all of the approaches (bode, nyquist, root locus).

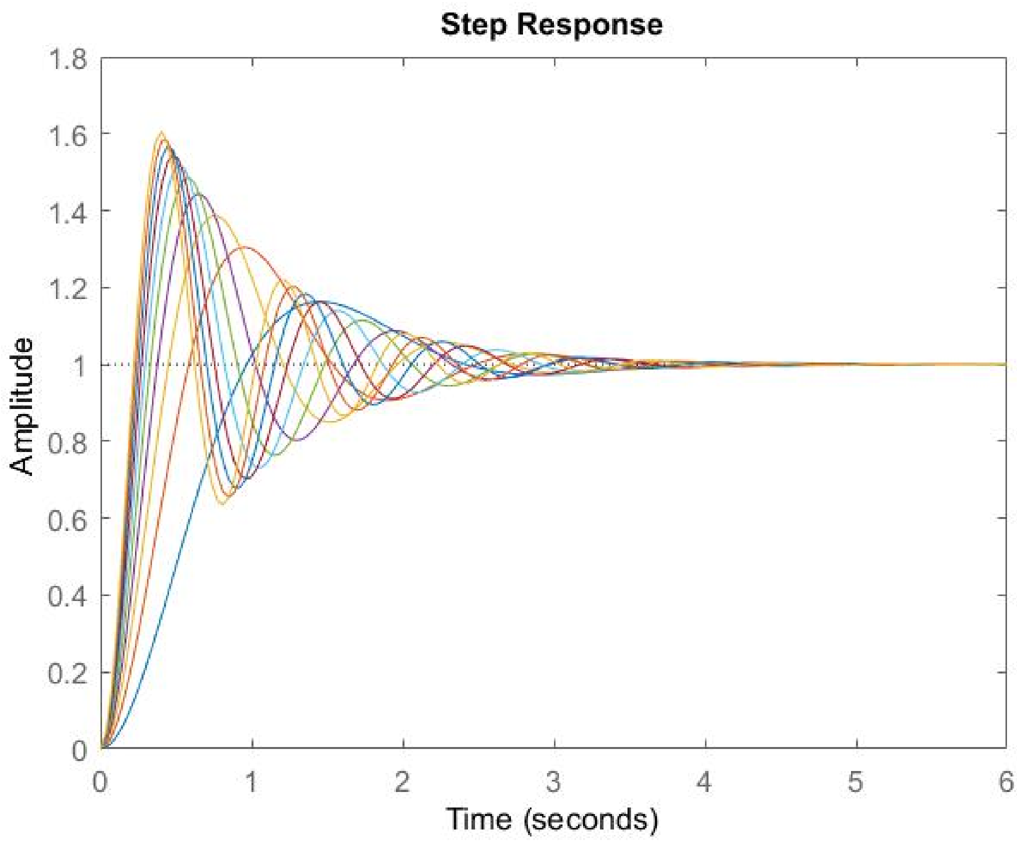
$$\begin{cases} bode: PM > 0 \ , GM \ \rightarrow +\infty \\ nyquist: P = 0 \, , N = 0 \ \rightarrow P = N \end{cases}$$

However, from the last plot it is obvious that system can follow the step input signal by having only P controller. Therefore, steady-state error in this condition is almost zero.

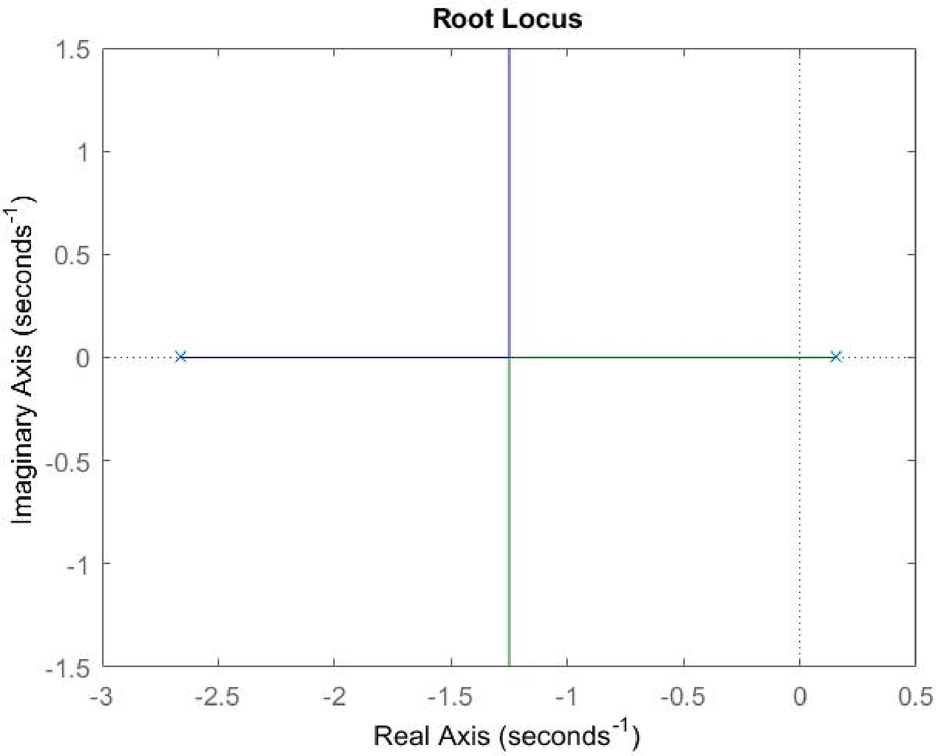
<u>Effect of increasing values of gain:</u> Increasing the gain of the controller from 100 to 1000, and seeing the results.

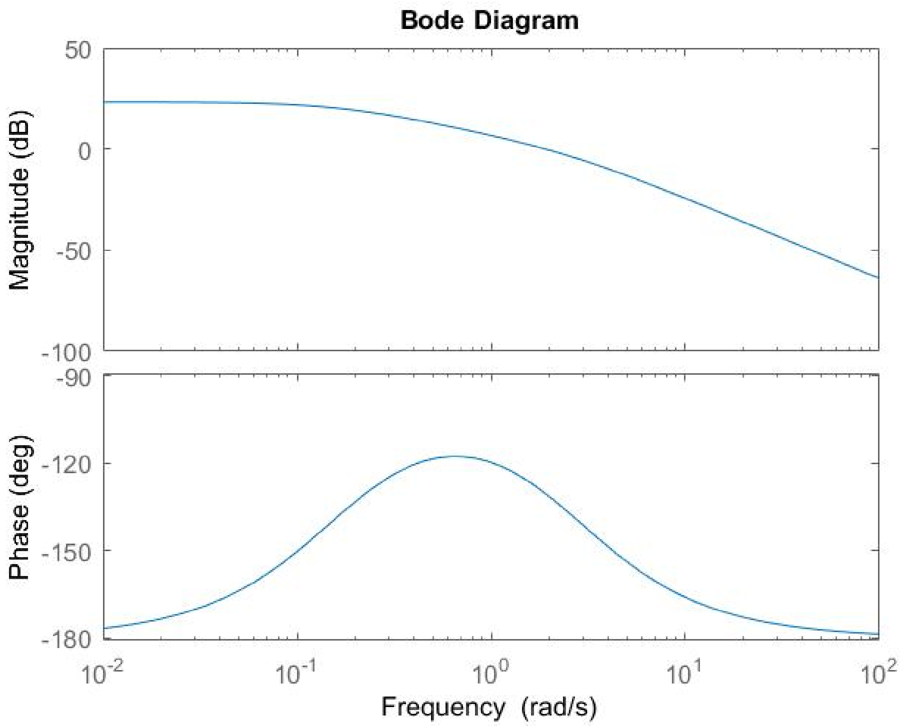


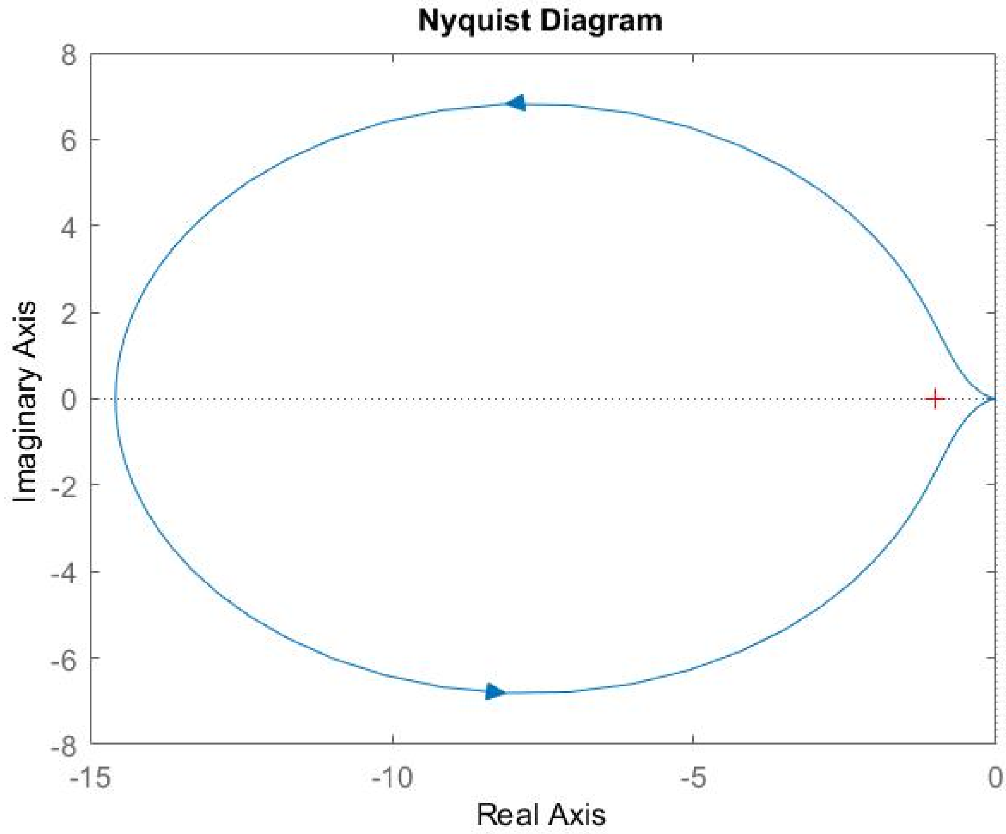

Nyquist Diagram
Imaginary Axis
Real Axis
200
150
100
50
0
-50
-100
-150
-200
-10
-8
-6
-4
-2
0

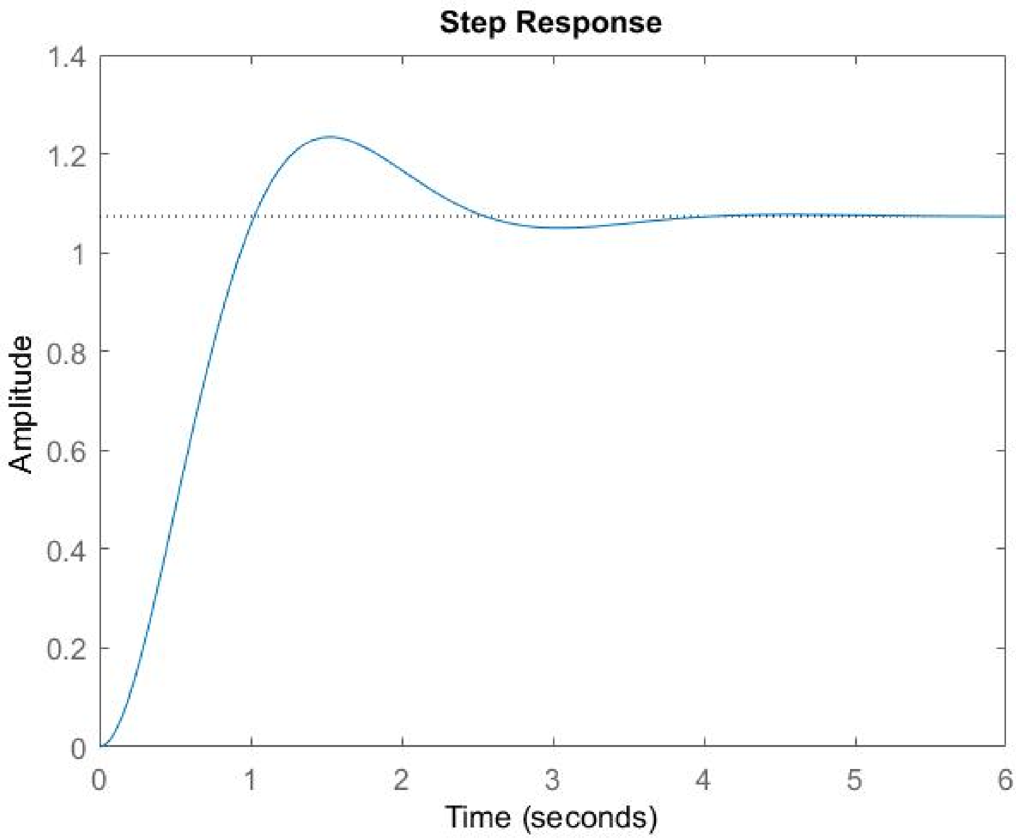

Step Response
Amplitude
Time (seconds)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6

- Case C) We consider k_3 as the stiffness of the system. Here are the outcomes and responses of the system both in time and Laplace domain resulted by Matlab (we assume $k_p = 100$, and then see the effect of increasing gain): NOTE: The bode criterion cannot be applied in this case because of positive pole.






Nyquist Diagram
Imaginary Axis
Real Axis
8
6
4
2
0
-2
-4
-6
-8
-15
-10
-5
0

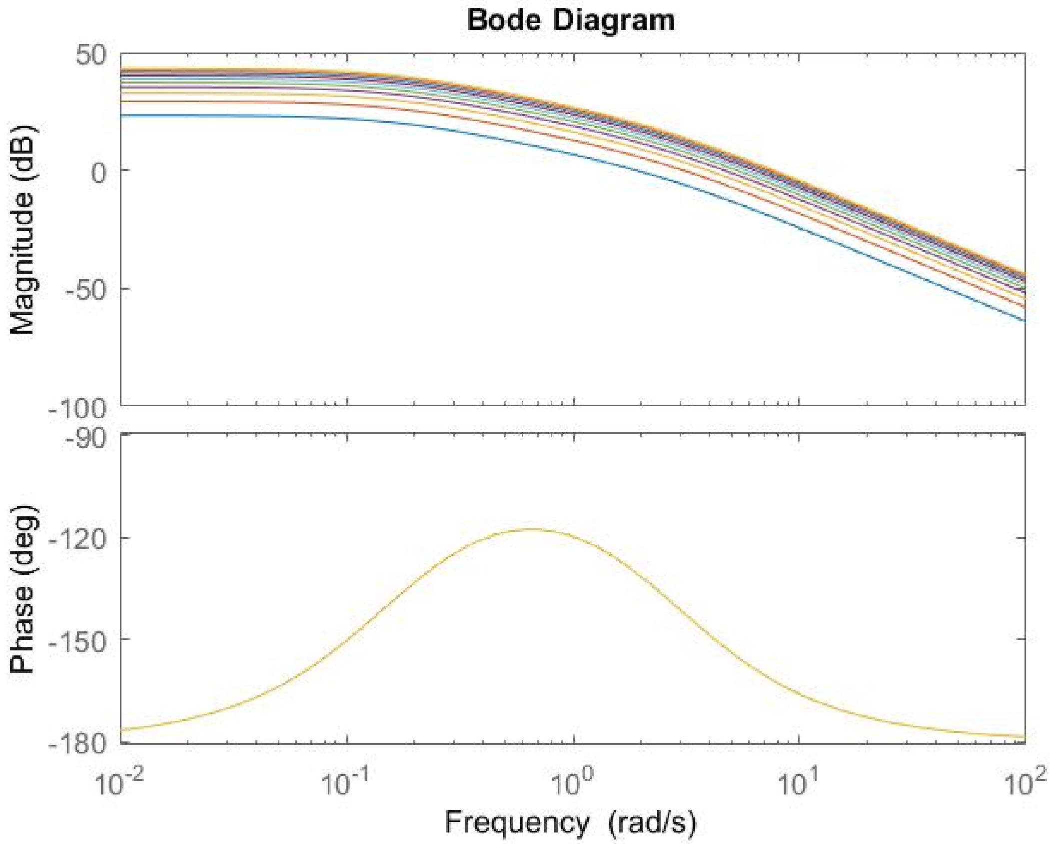

Step Response
Amplitude
Time (seconds)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6

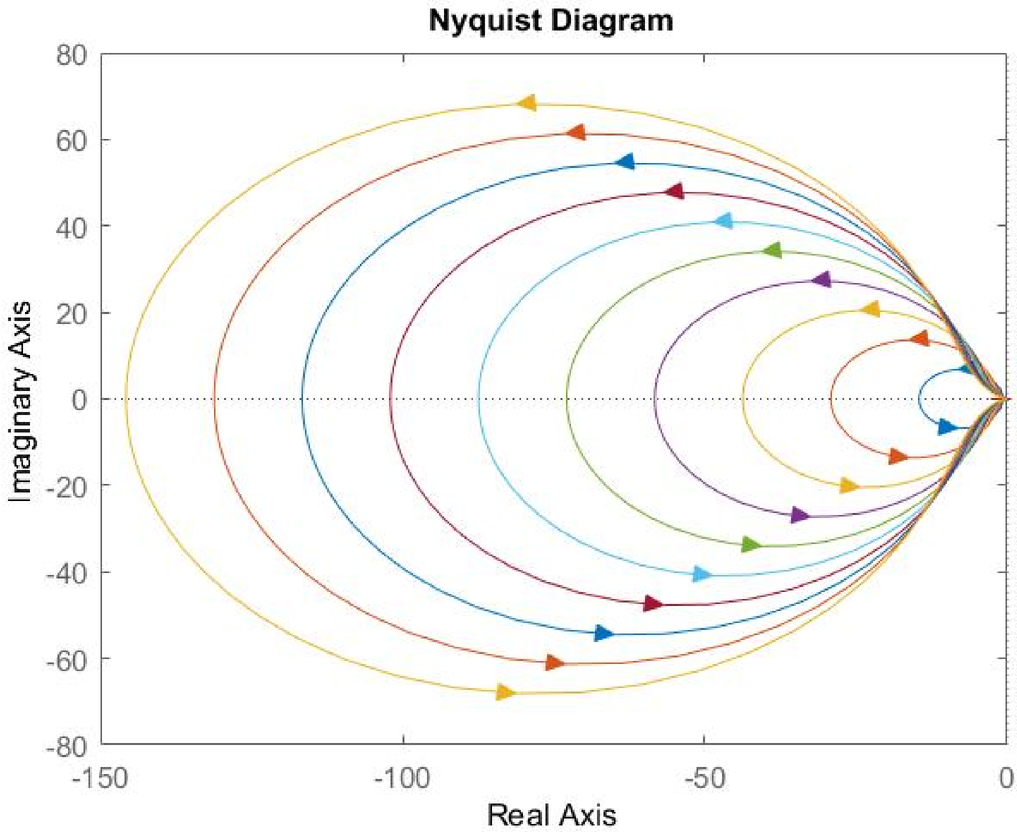
As we see the results, system is stable for **high** values of k_p, and stability criterion is satisfied in these approaches (nyquist, root locus).

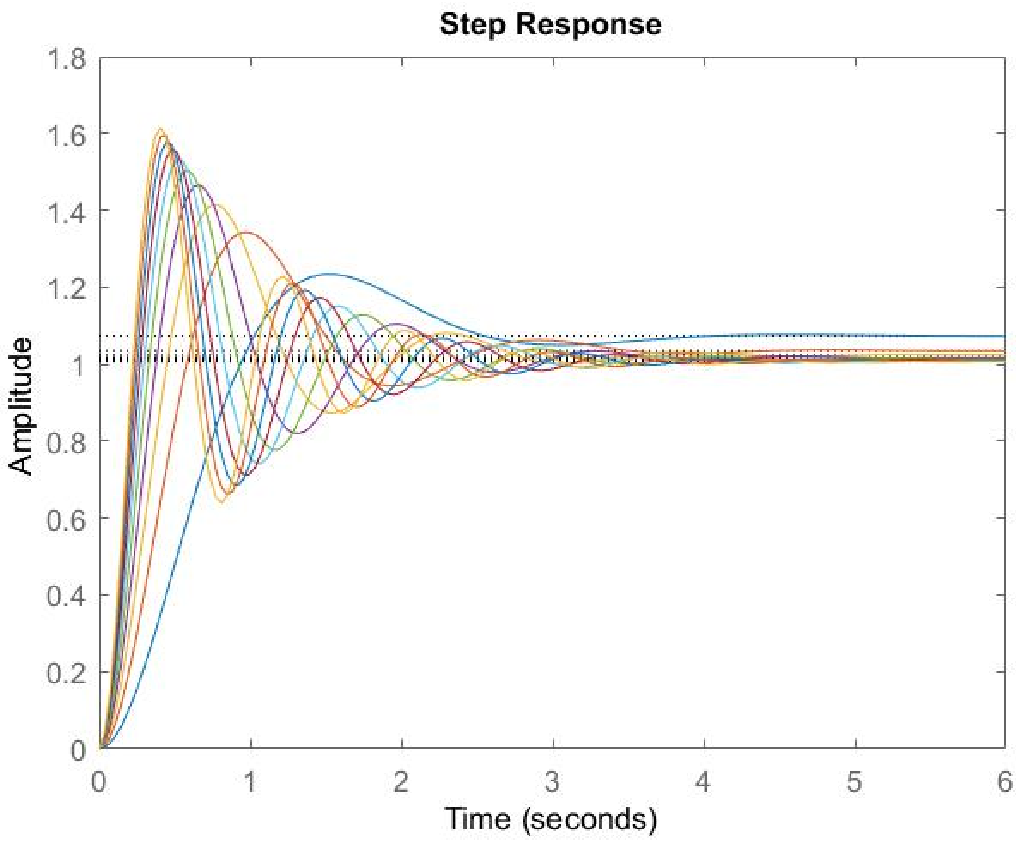
$nyquist: P = 1\ , N = 1\ \rightarrow P = N$

Also, from the last plot it is obvious that system can follow the step input signal by having only P controller. Therefore, steady-state error in this condition is very low. ($e_{ss} = \theta_{\infty} - \theta_{ref} \cong 1.07 - 1 = 0.07$)

Effect of increasing values of gain: Increasing the gain of the controller from 100 to 1000, and seeing the results.




Nyquist Diagram
80
60
40
20
0
-20
-40
-60
-80
Imaginary Axis
-150
-100
-50
0
Real Axis


Step Response
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
0
1
2
3
4
5
6
Time (seconds)

By increasing the gain, it is obvious that the system tends to keep its stability but by having a low gain there will be no loop around point -1. So, the system will be unstable in this case with K_p critical.

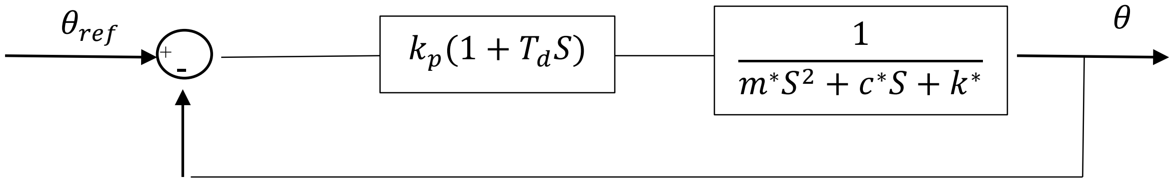
2) Proportional-Derivative Controller:

The equation for this type of controller acting on torque C is $C = k_p(\theta_{ref} - \theta) + k_d(\dot{\theta}_{ref} - \dot{\theta})$.

Equation and block diagram in Laplace domain are defined in this way:

$$(m^* S^2 + c^* S + k^*)\, \theta(S) = C(S)$$

$$C(S) = k_p(1 + T_d S)(\theta_{ref}(S) - \theta(S))$$

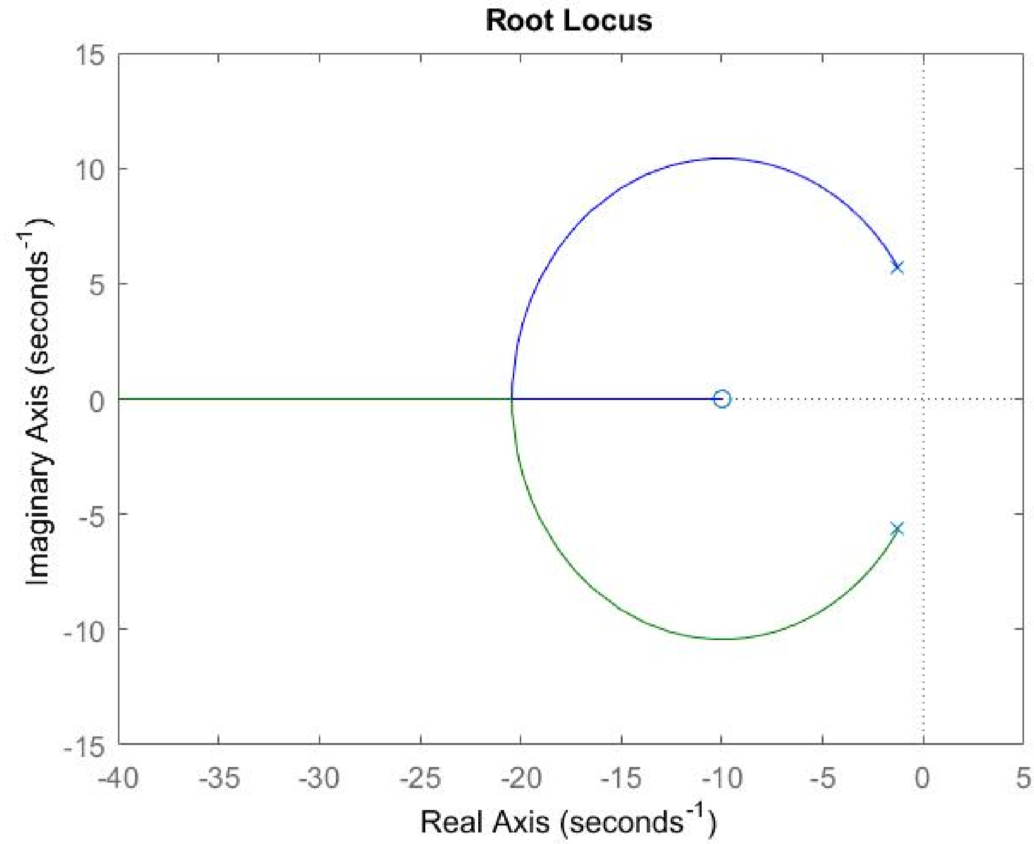


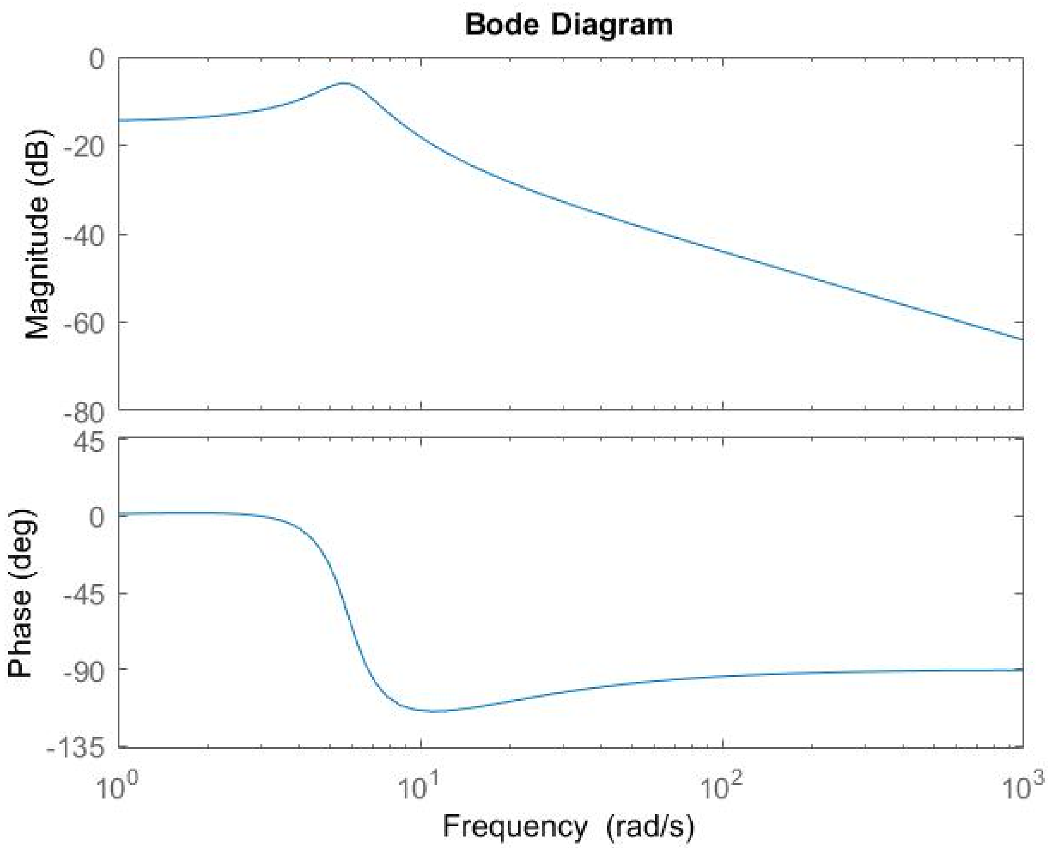
The open loop transfer function for analyzing stability is needed which is:

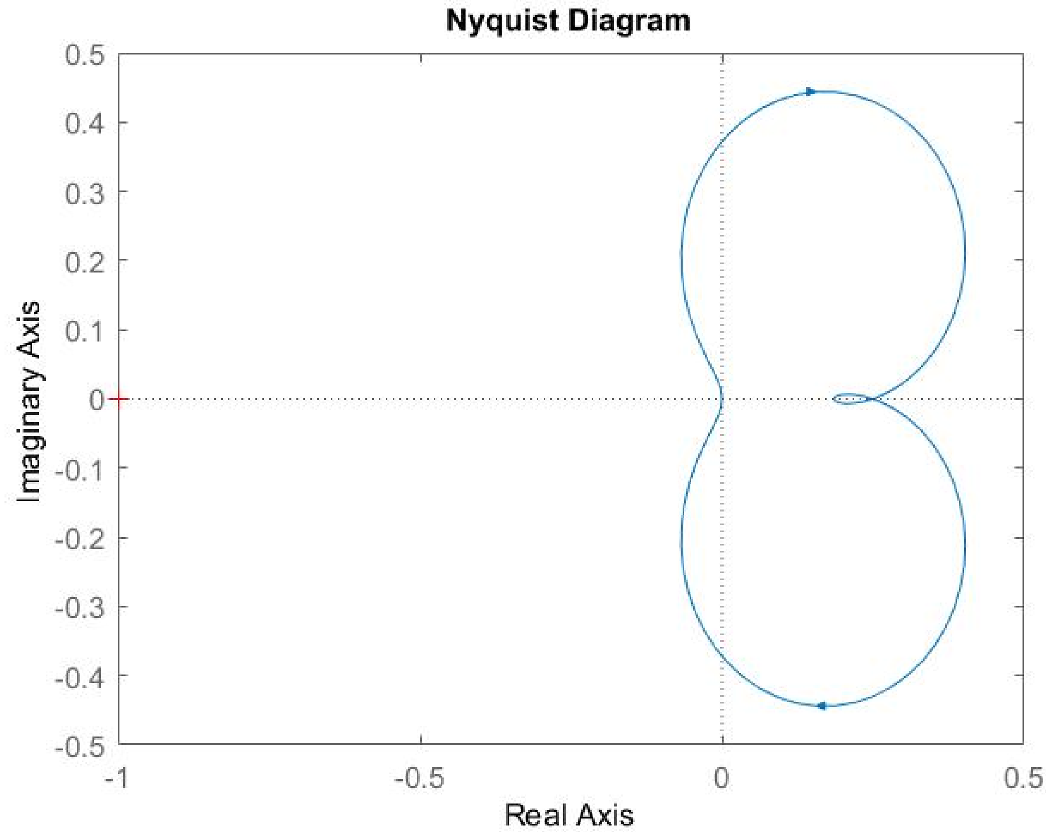
$$G(S) = \frac{k_p(1 + T_d S)}{(m^* S^2 + c^* S + k^*)}$$

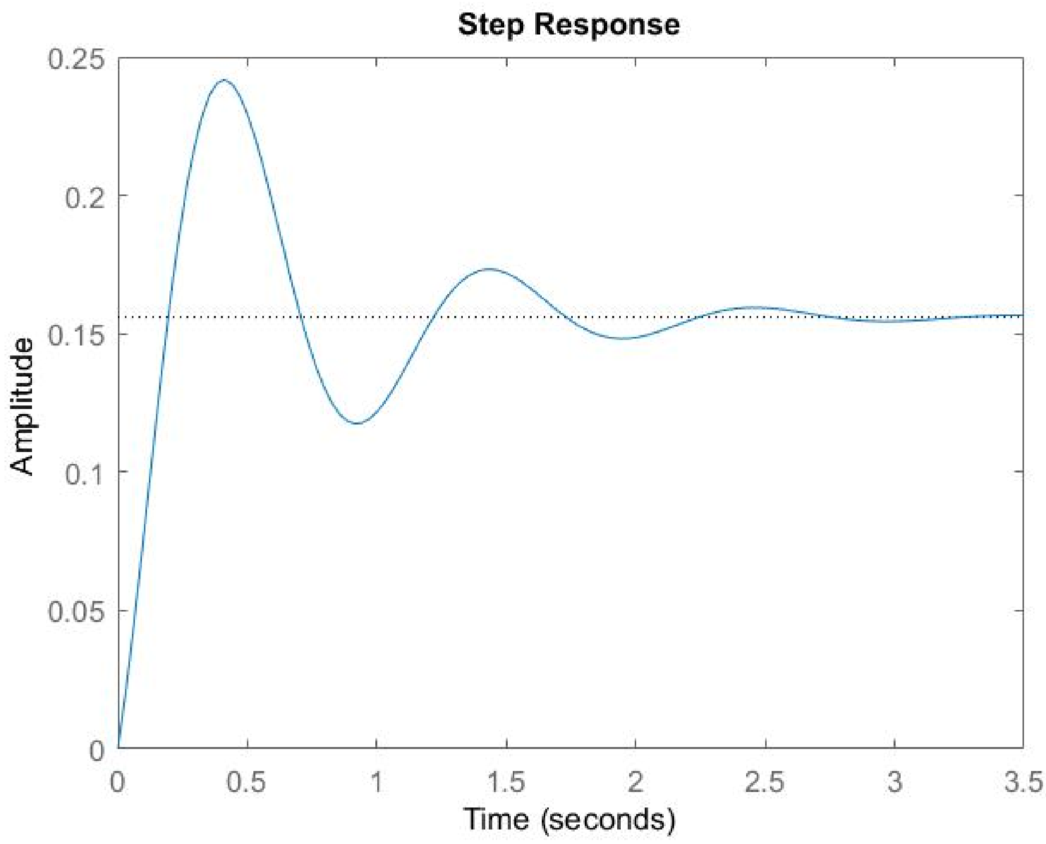
In this problem, We consider k_1 as the stiffness of the system. We have to investigate two cases in which $\left|z = \frac{1}{T_d}\right| < \left|P_{1,2}\right|$ or $\left|z = \frac{1}{T_d}\right| > \left|P_{1,2}\right|$

a) $\left|z = \frac{1}{T_d}\right| > \left|P_{1,2}\right|$ by choosing $T_d = 0.1$


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
15
10
5
0
-5
-10
-15
-40
-35
-30
-25
-20
-15
-10
-5
0
5


Bode Diagram
Magnitude (dB)
0
-20
-40
-60
-80
Phase (deg)
45
0
-45
-90
-135
10^0
10^1
10^2
10^3
Frequency (rad/s)


Nyquist Diagram
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Imaginary Axis
-1
-0.5
0
0.5
Real Axis

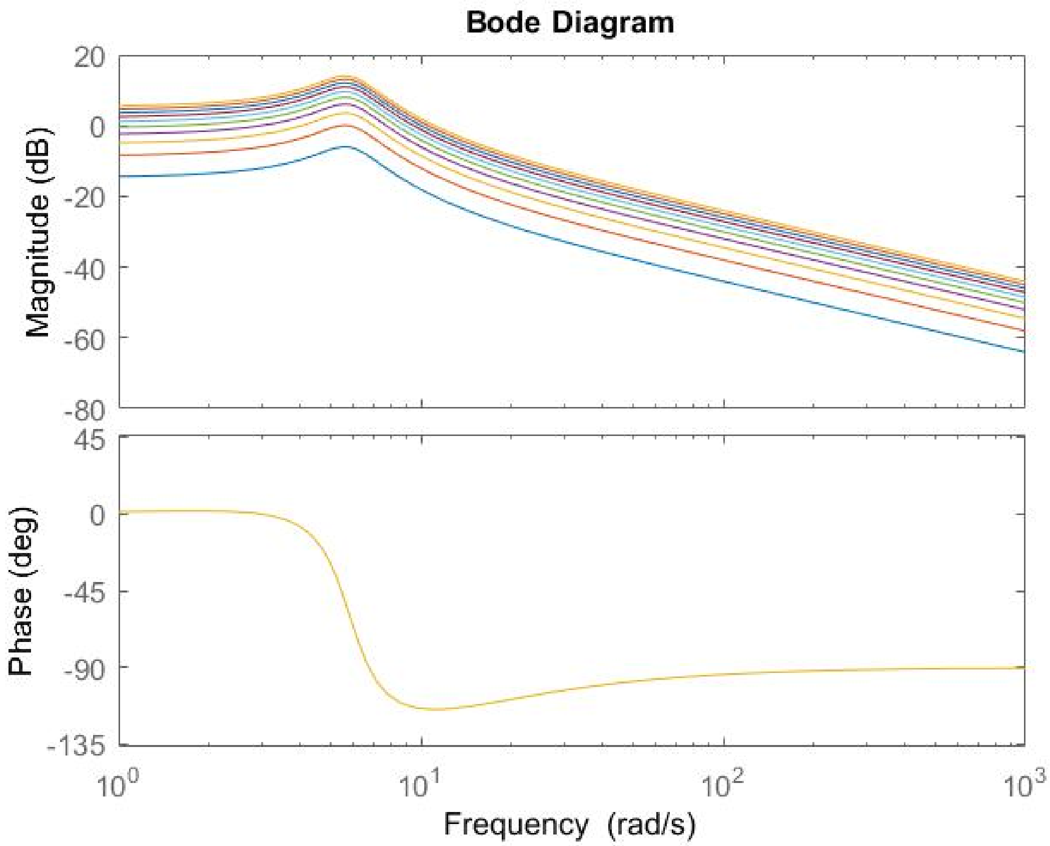

Step Response
0.25
0.2
0.15
0.1
0.05
0
Amplitude
0
0.5
1
1.5
2
2.5
3
3.5
Time (seconds)

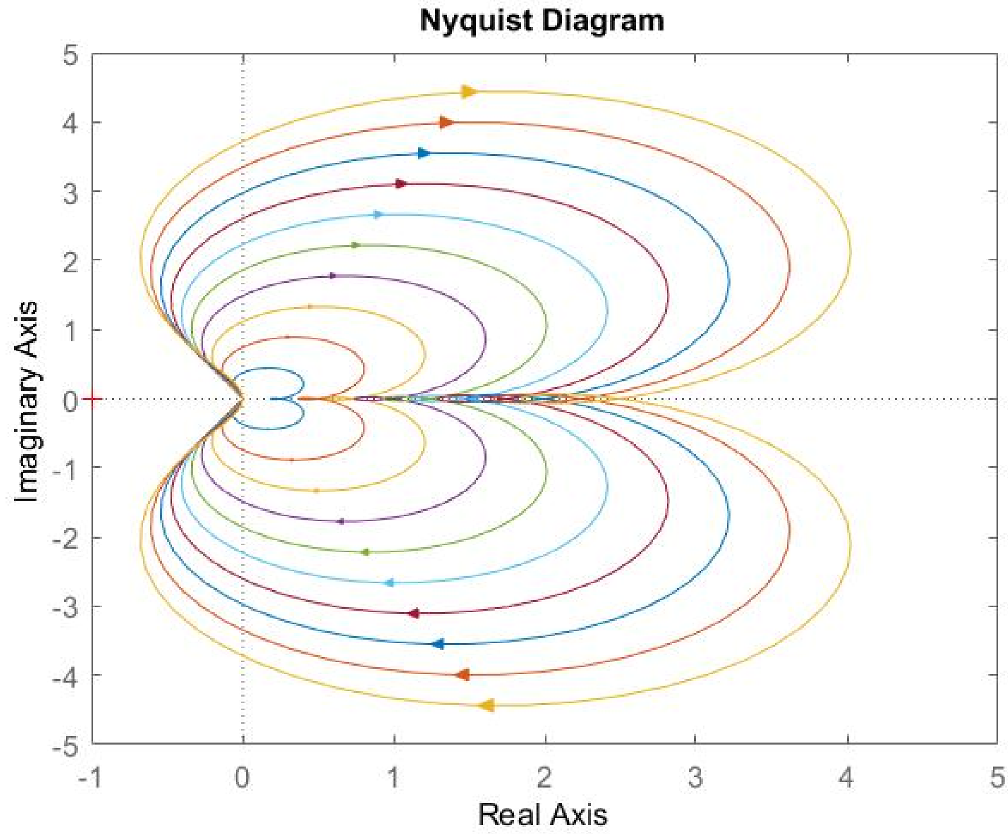
As we see the results, system is always stable for any values of k_p, and stability criterion is satisfied in these approaches (bode, nyquist, root locus).

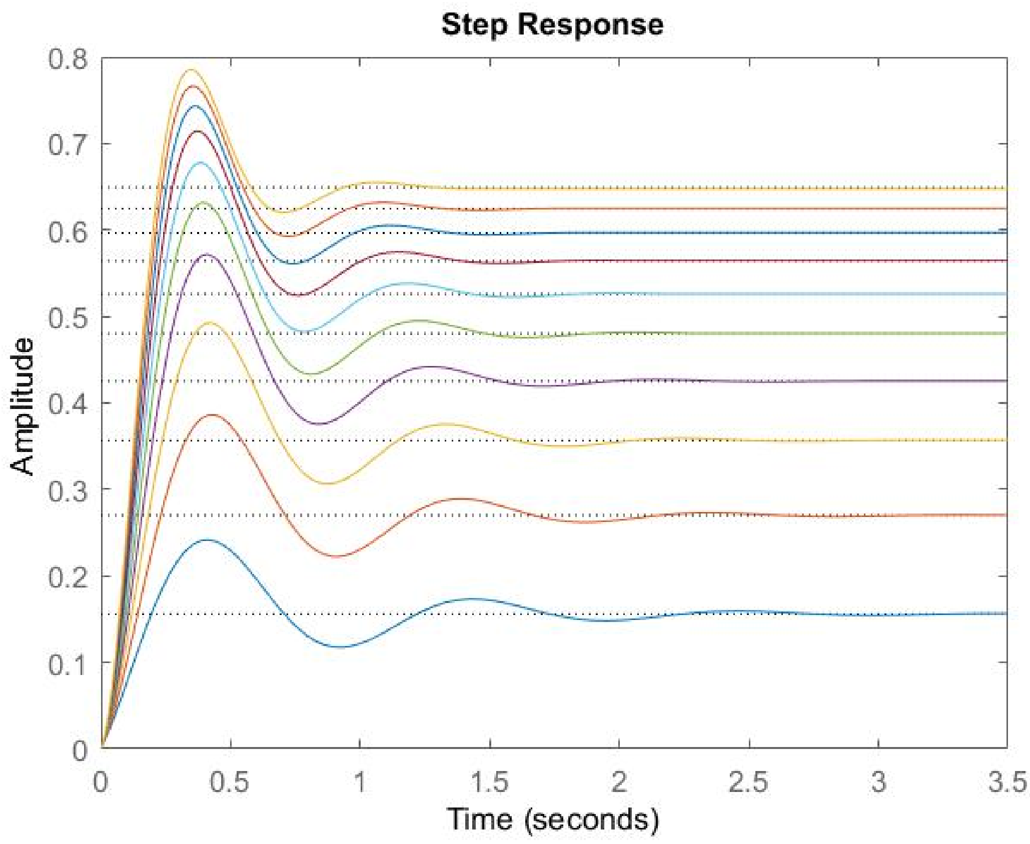
$$\begin{cases} bode: PM > 0\ , GM\ \rightarrow +\infty \\ nyquist: P = 0\ , N = 0\ \rightarrow P = N \end{cases}$$

Also, from the last plot it is obvious that system cannot follow the step input signal by having PD controller. Therefore, steady-state error in this condition is very high. ($e_{ss} = \theta_\infty - \theta_{ref} \cong 1 - 0.16 = 0.84$)

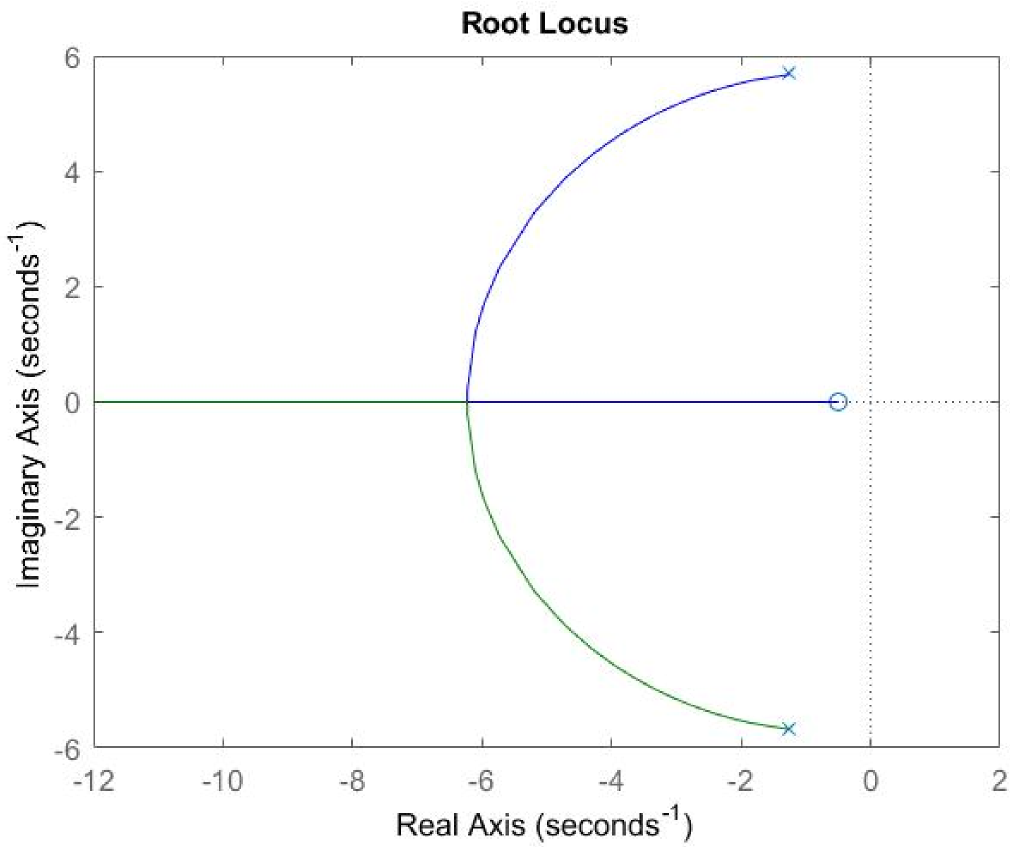
<u>Effect of increasing values of gain:</u> Increasing the gain of the controller from 100 to 1000, and seeing the results.

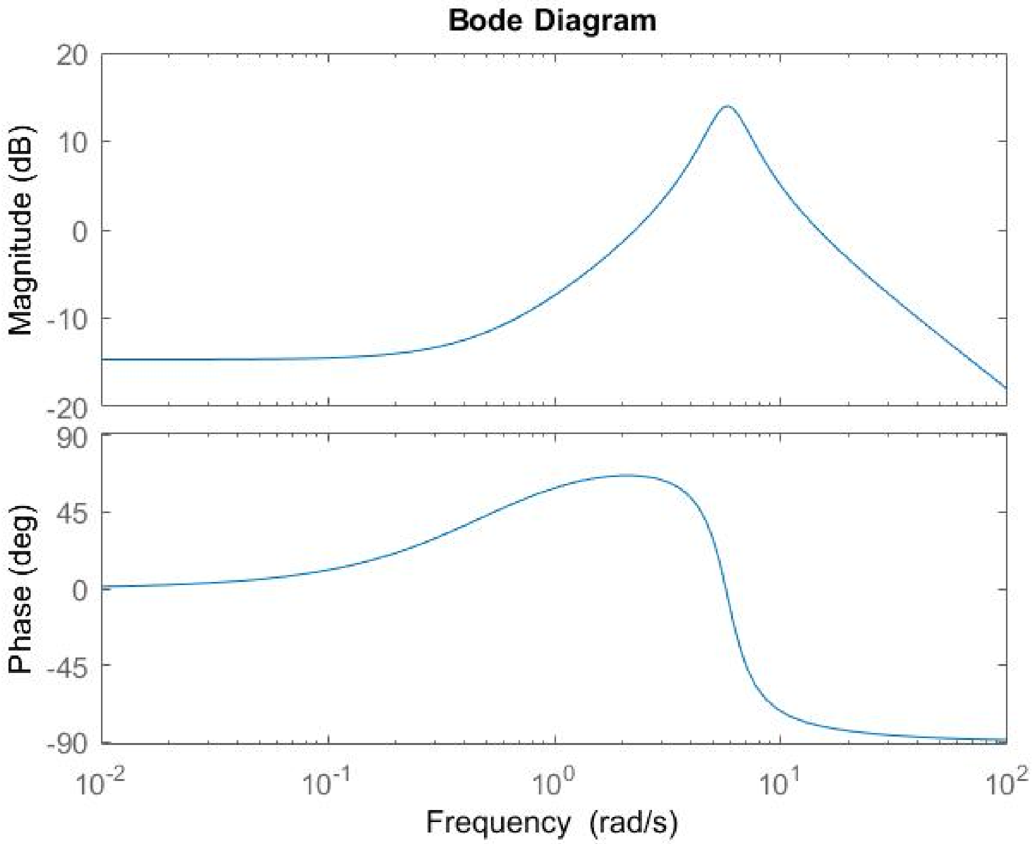


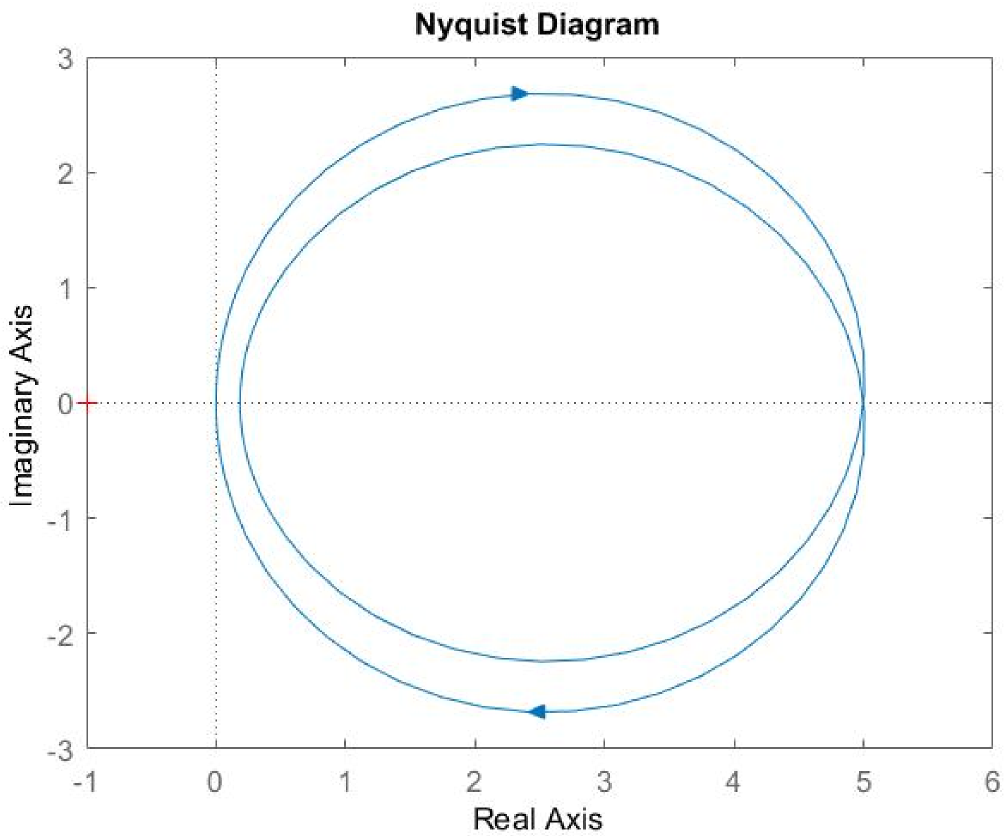

Nyquist Diagram
Imaginary Axis
Real Axis
5
4
3
2
1
0
-1
-2
-3
-4
-5
-1
0
1
2
3
4
5

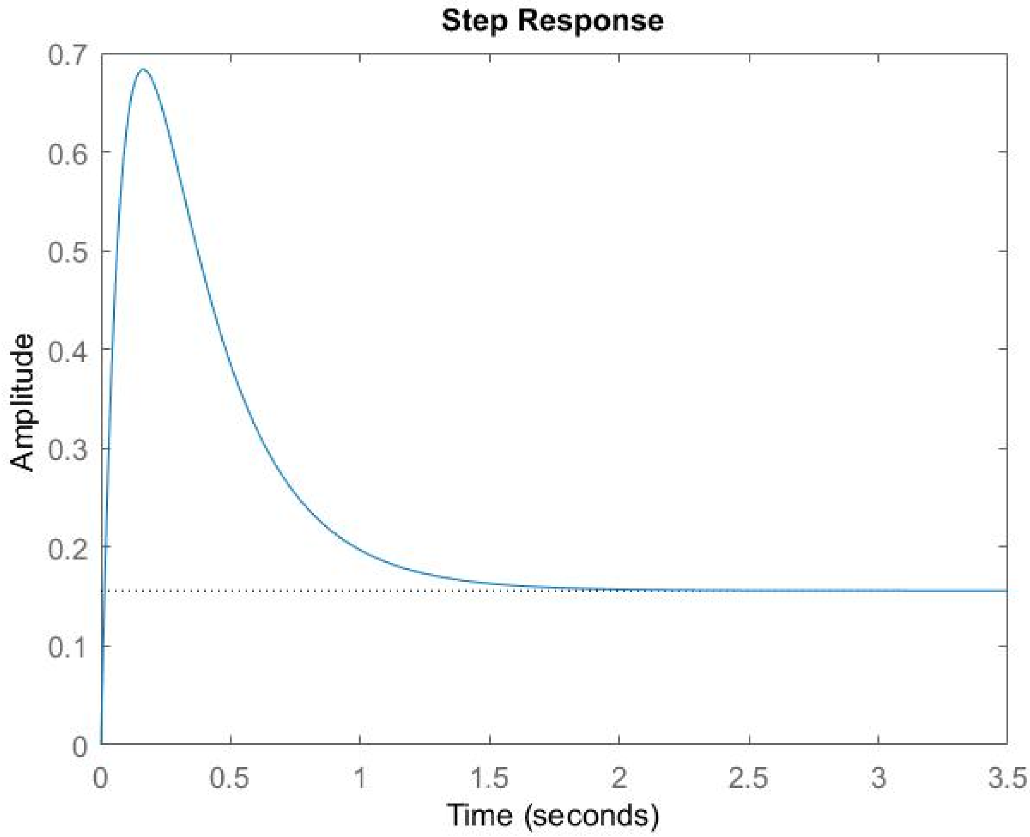

Step Response
Amplitude
Time (seconds)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5

b) $\left|z = \frac{1}{T_d}\right| < \left|P_{1,2}\right|$ by choosing $T_d = 2$




Bode Diagram
Magnitude (dB)
20
10
0
-10
-20
Phase (deg)
90
45
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)


Nyquist Diagram
Imaginary Axis
3
2
1
0
-1
-2
-3
-1
0
1
2
3
4
5
6
Real Axis

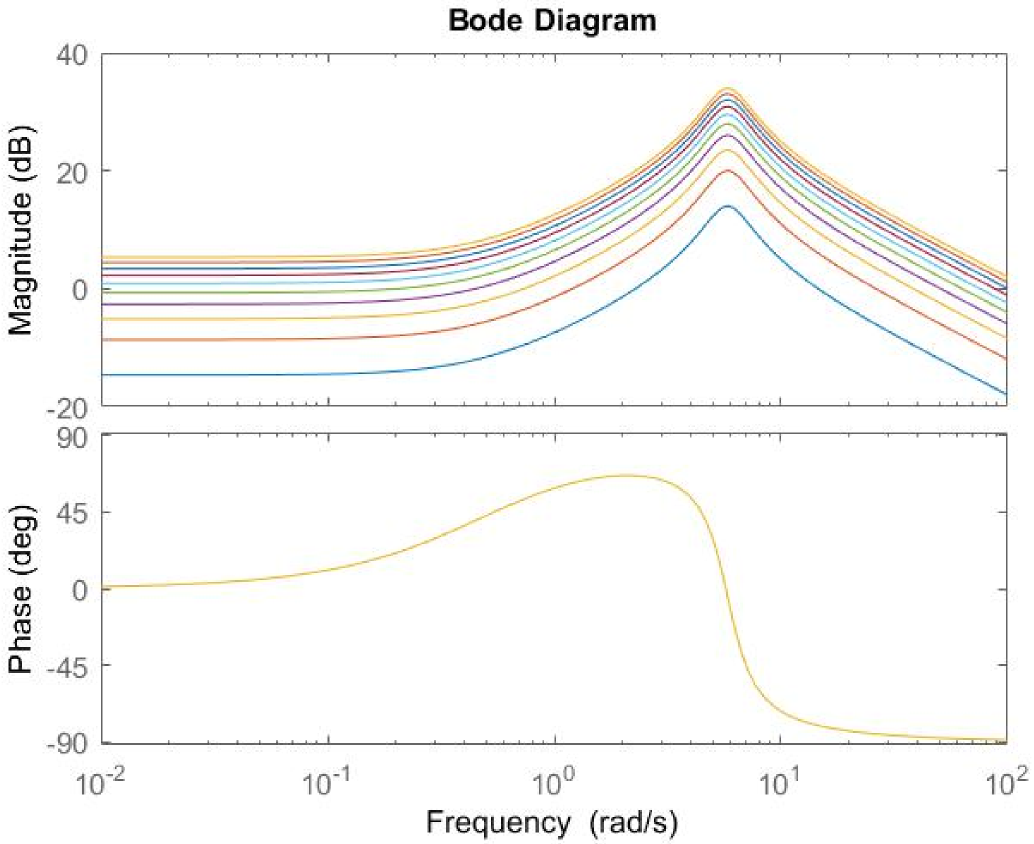


As we see the results, system is always stable for any values of k_p, and stability criterion is satisfied in these approaches (bode, nyquist, root locus).

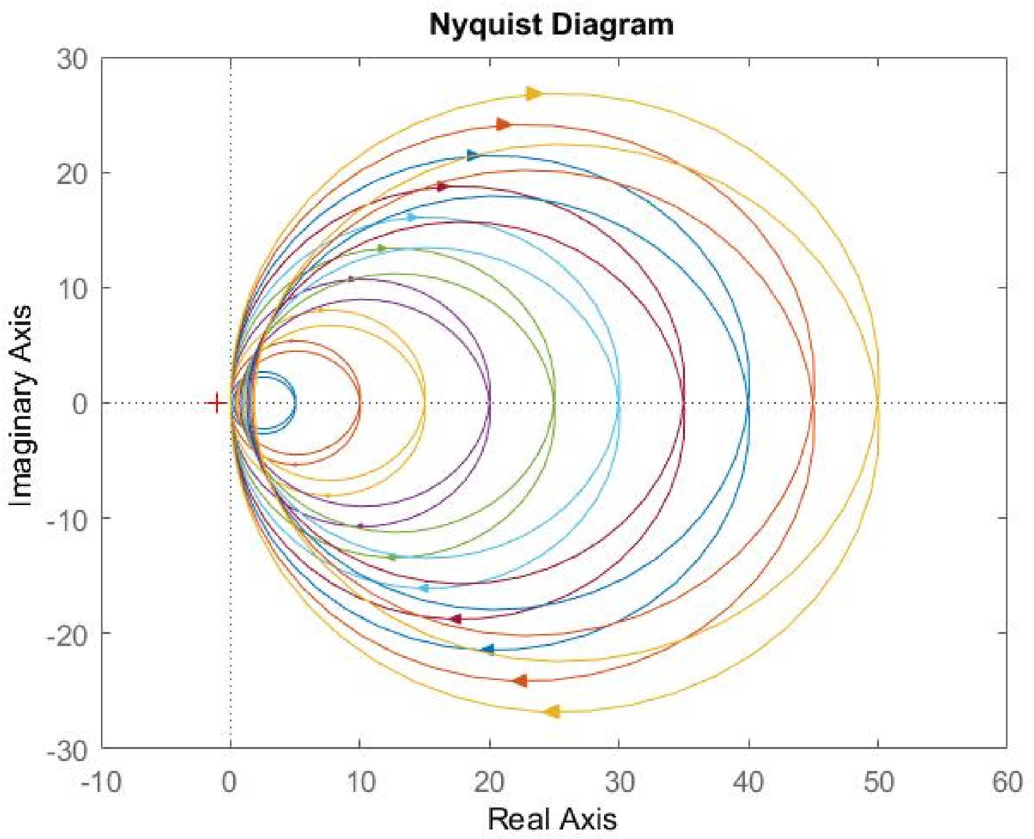
$$\begin{cases} bode: PM > 0 \ , GM \ \rightarrow +\infty \\ nyquist: P = 0 \, , N = 0 \ \rightarrow P = N \end{cases}$$

Also, from the last plot it is obvious that system cannot follow the step input signal by having only P controller. Therefore, steady-state error in this condition is very high. ($e_{ss} = \theta_\infty - \theta_{ref} \cong 1 - 0.16 = 0.84$)

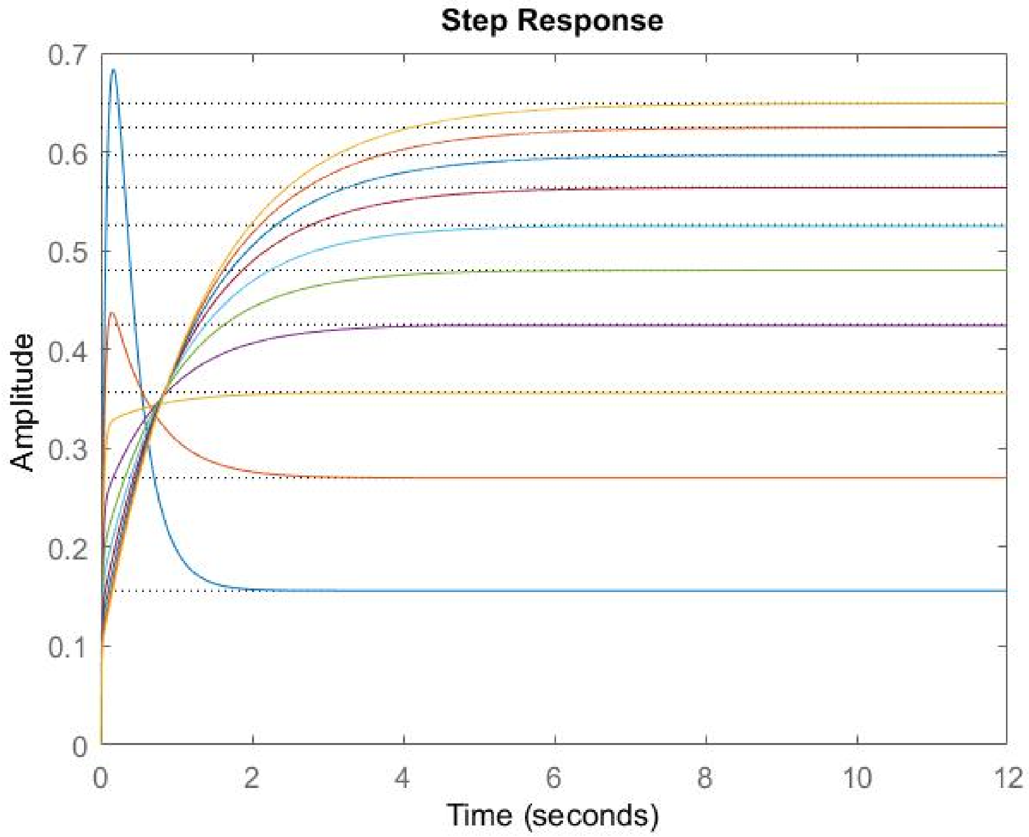
<u>Effect of increasing values of gain:</u> Increasing the gain of the controller from 100 to 1000, and seeing the results.



Bode Diagram
Magnitude (dB)
Phase (deg)
40
20
0
-20
90
45
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)



Nyquist Diagram
Imaginary Axis
Real Axis
30
20
10
0
-10
-20
-30
-10
0
10
20
30
40
50
60



Step Response
Amplitude
Time (seconds)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12

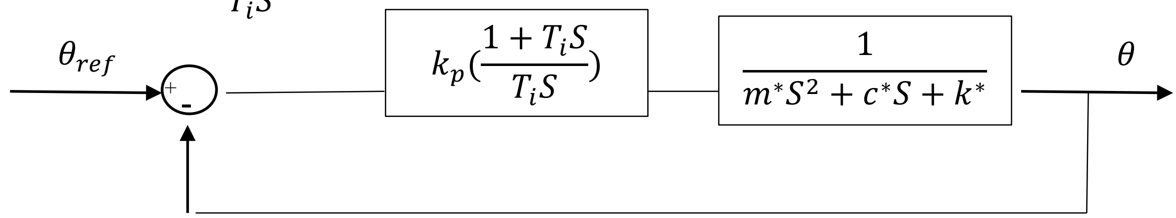
3) Proportional-Integrative Controller:

The equation for this type of controller acting on torque C is $C = k_p(\theta_{ref} - \theta) + k_i(\int \theta_{ref}\, dt\ - \int \theta\, dt\ )$.

Equation and block diagram in Laplace domain are defined in this way:

$(m^* S^2 + c^* S + k^*)\ \theta(S) = C(S)$

$C(S) = k_p(1 + \frac{1}{T_i S})(\theta_{ref}(S) - \theta(S))$

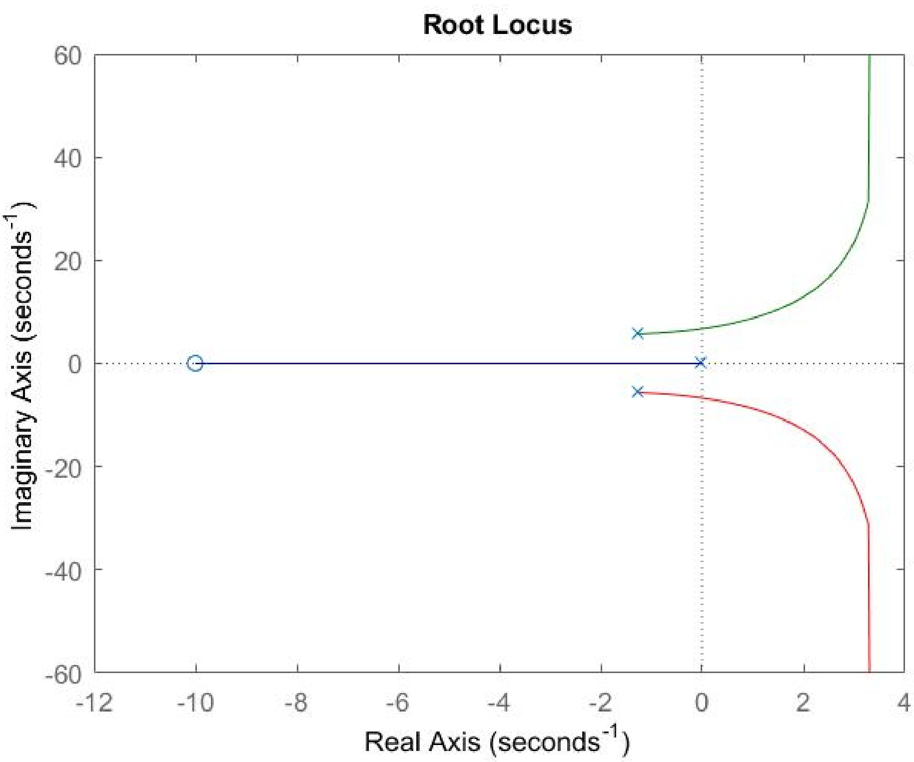


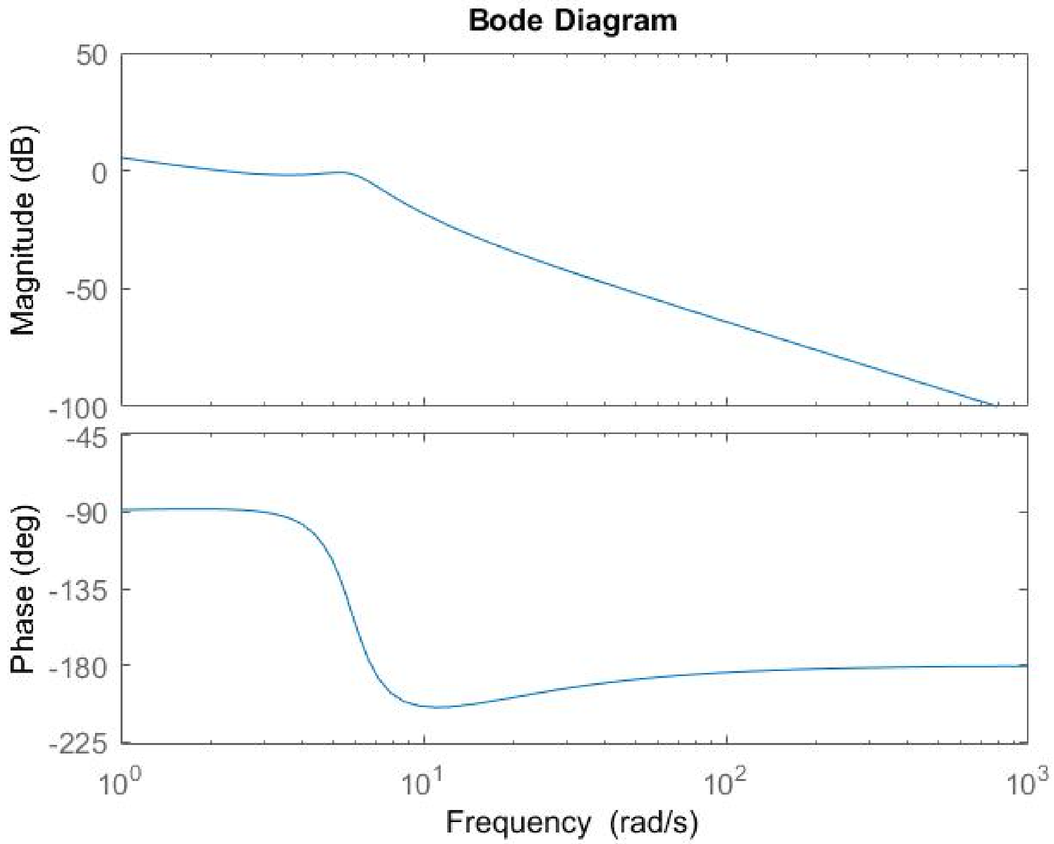
The open loop transfer function for analyzing stability is needed which is:

$$G(S) = \frac{k_p(1 + T_i S)}{T_i S(m^* S^2 + c^* S + k^*)}$$

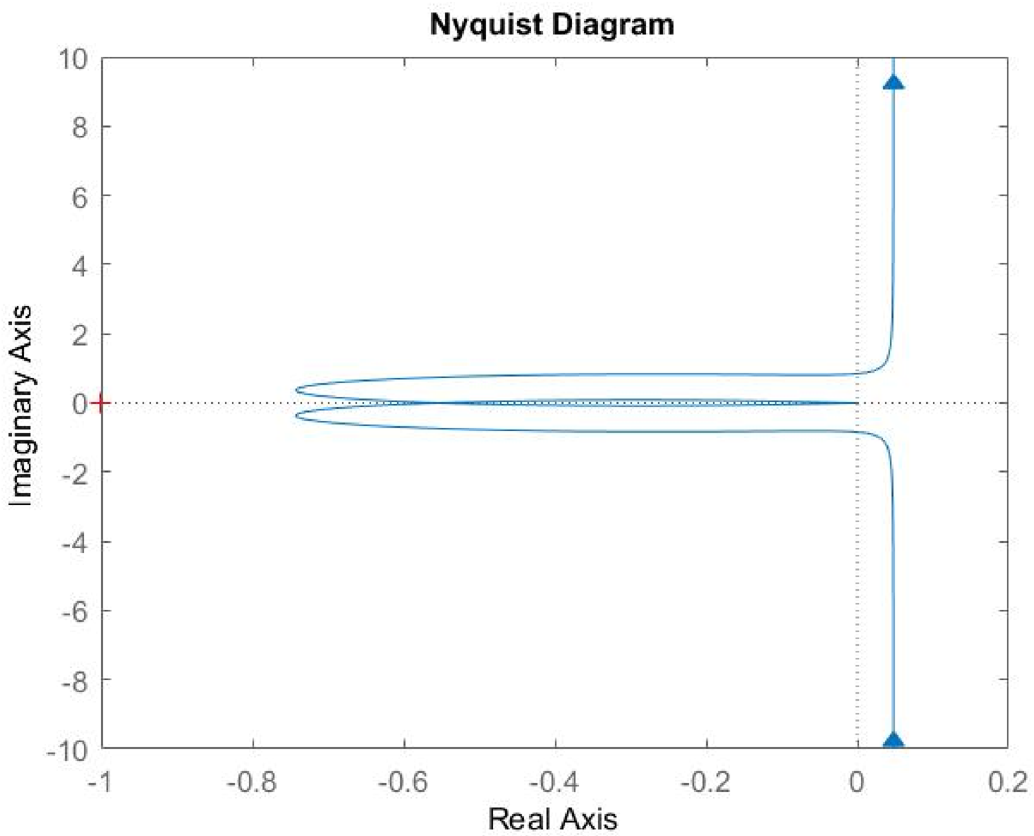
In this problem, We consider k_1 as the stiffness of the system. We have to investigate two cases in which $\left|z = \frac{1}{T_i}\right| < \left|P_{1,2}\right|\ or\ \left|z = \frac{1}{T_i}\right| > \left|P_{1,2}\right|$

a) $\left|z = \frac{1}{T_i}\right| > \left|P_{1,2}\right|$ by choosing $T_i = 0.1$

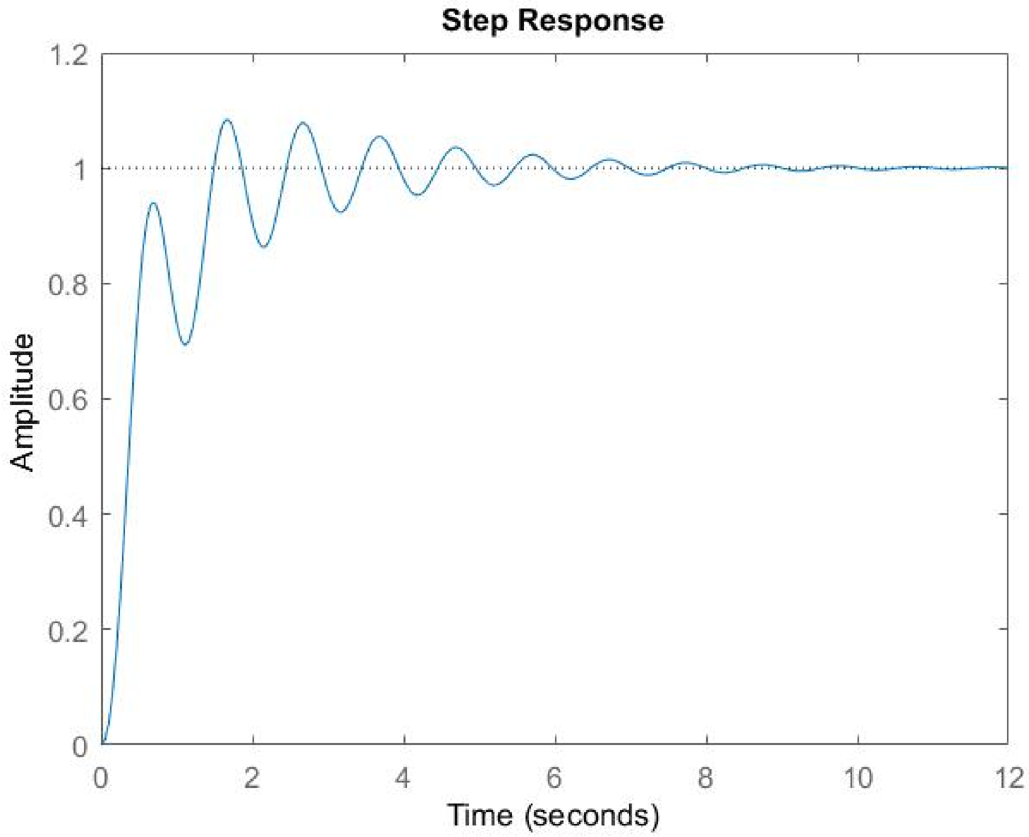




Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
50
0
-50
-100
-45
-90
-135
-180
-225
10^0
10^1
10^2
10^3



Nyquist Diagram
Imaginary Axis
Real Axis
10
8
6
4
2
0
-2
-4
-6
-8
-10
-1
-0.8
-0.6
-0.4
-0.2
0
0.2

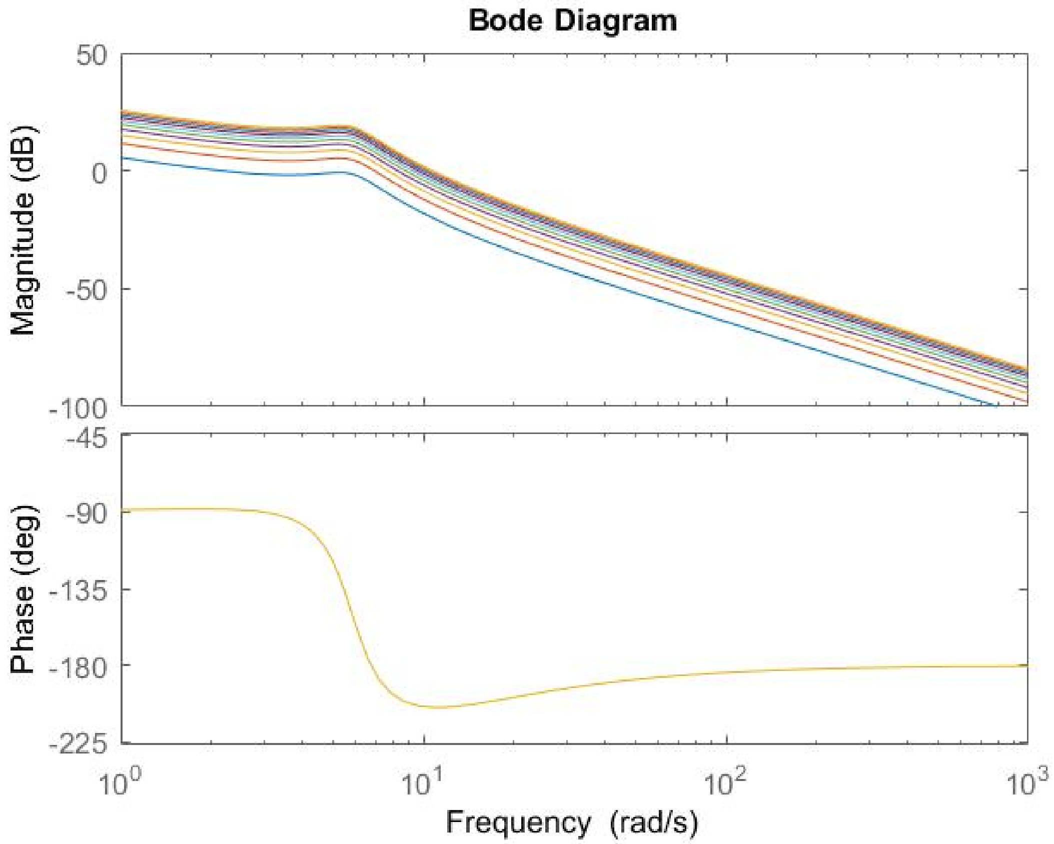


As we see the results, system is only stable for low values of k_p , and stability criterion is not satisfied in these approaches with high values of k_p (bode, nyquist, root locus).

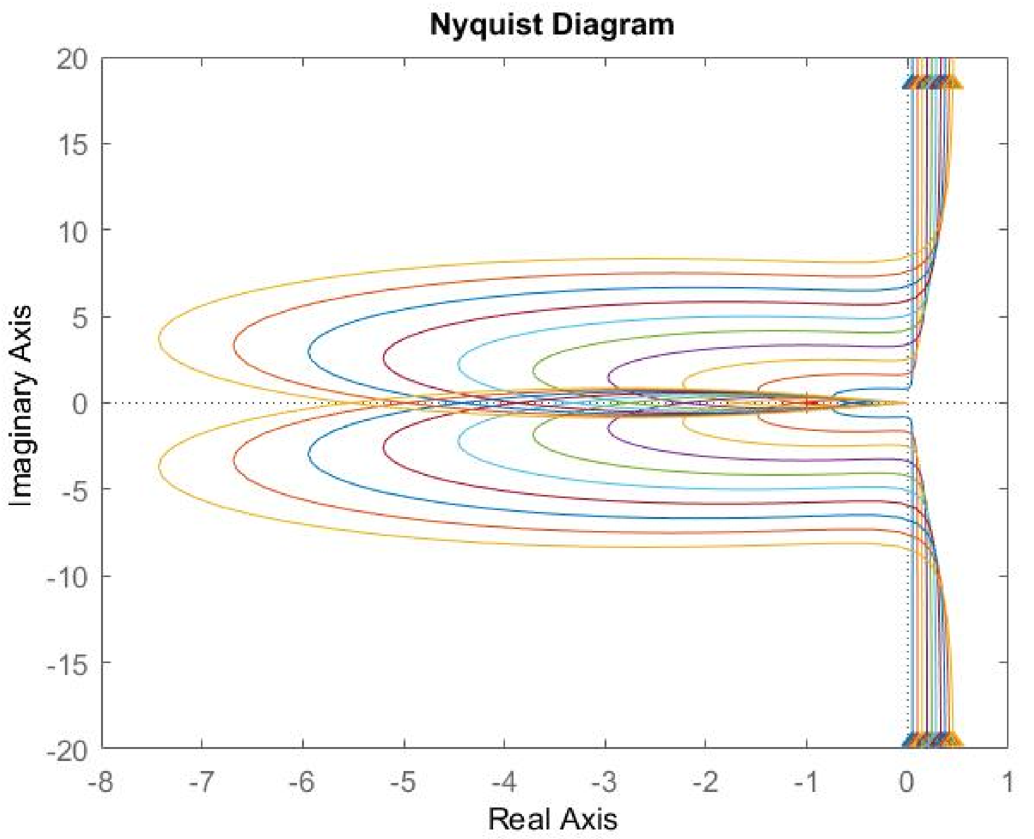
$$\begin{cases} bode: PM < 0 \ , GM > 0 \\ nyquist: P = 0 \, , N = 1 \ \rightarrow P \neq N \end{cases}$$

Also, from the last plot it is obvious that system can follow the step input signal by having PI controller. Therefore, steady-state error in this condition is almost zero.

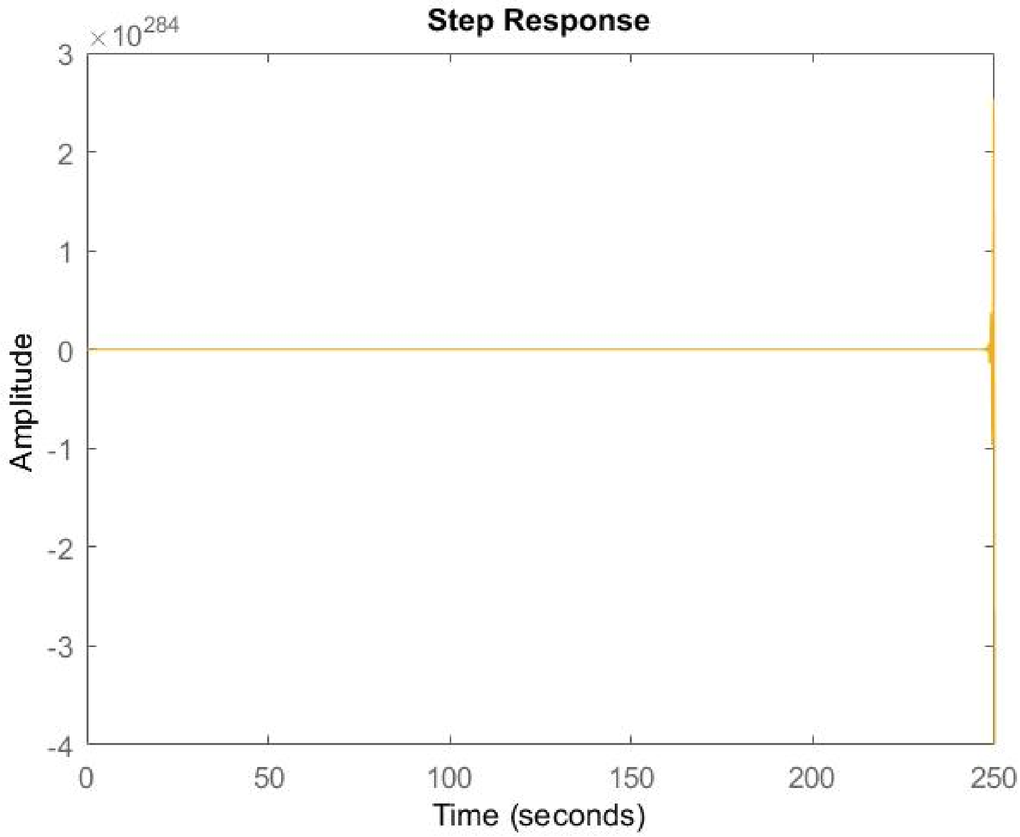
<u>Effect of increasing values of gain:</u> Increasing the gain of the controller from 100 to 1000, and seeing the results.



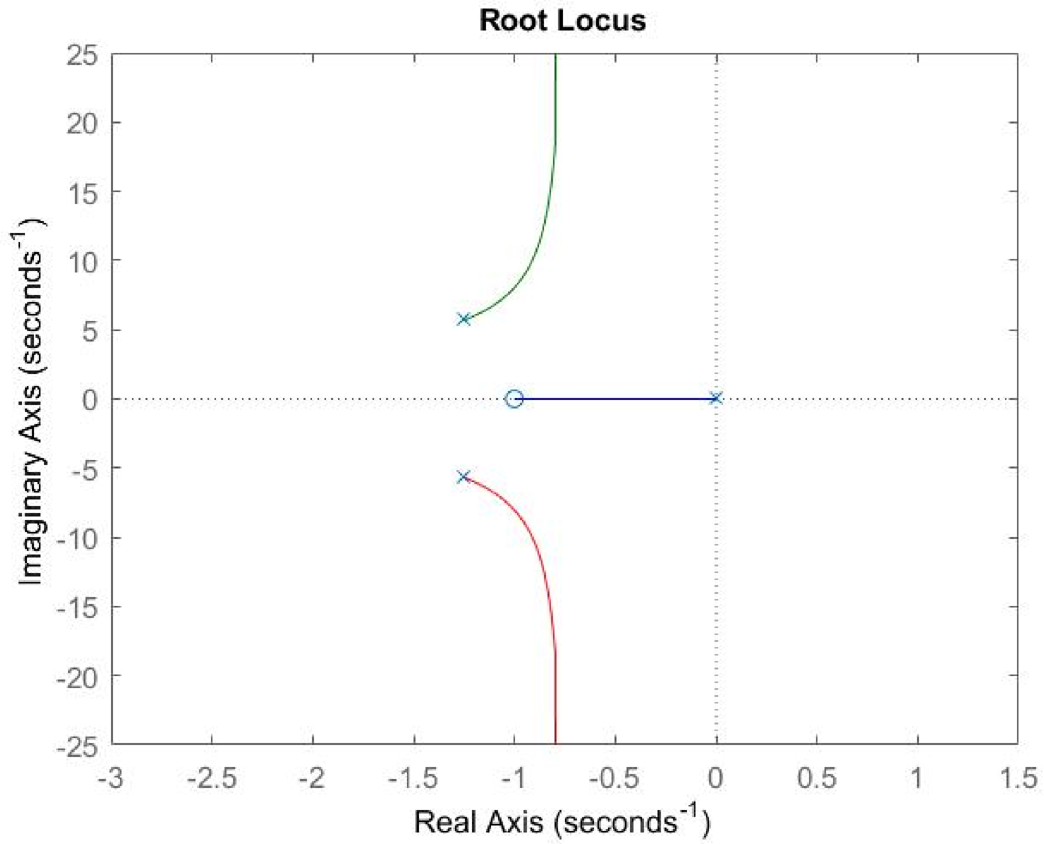
Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-45
-90
-135
-180
-225
10^0
10^1
10^2
10^3
Frequency (rad/s)

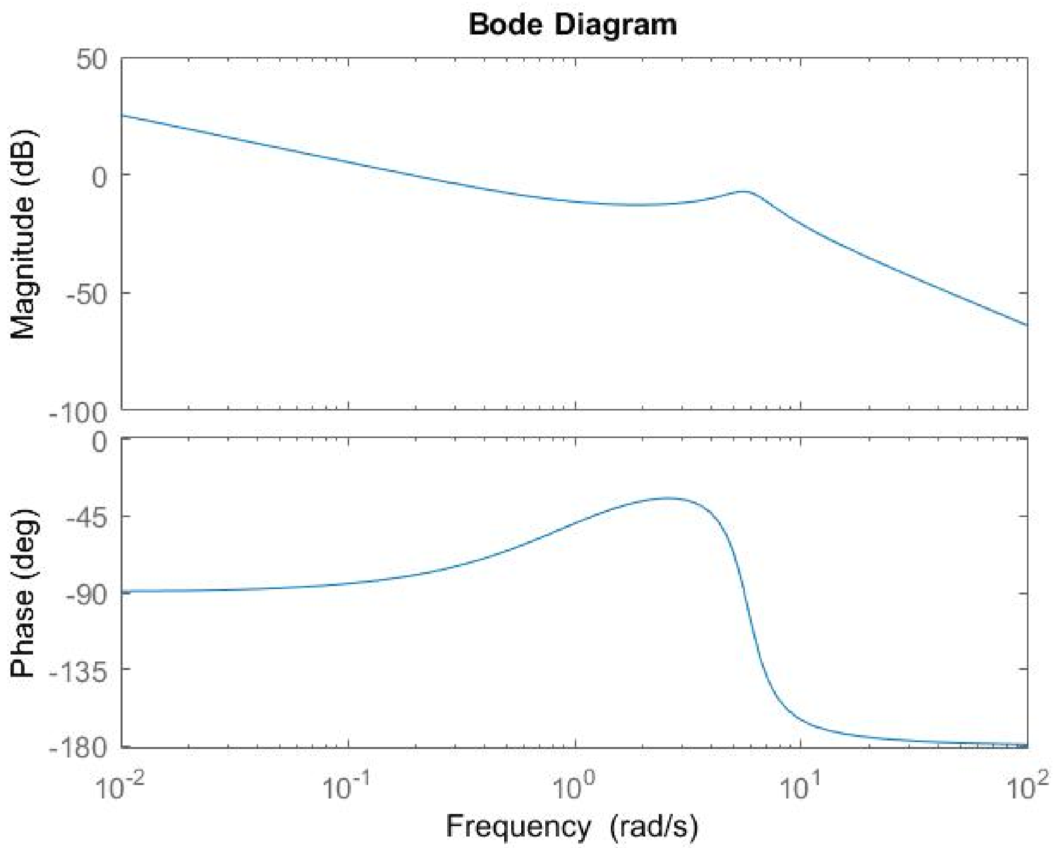


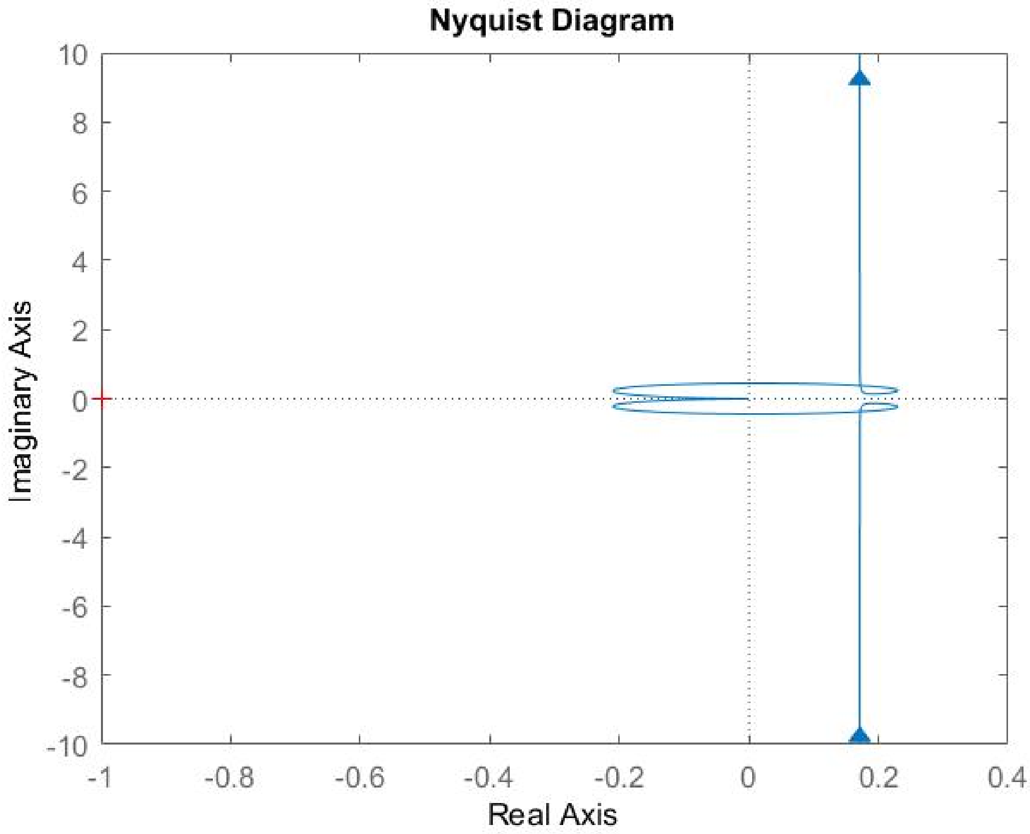
Nyquist Diagram
Imaginary Axis
20
15
10
5
0
-5
-10
-15
-20
-8
-7
-6
-5
-4
-3
-2
-1
0
1
Real Axis

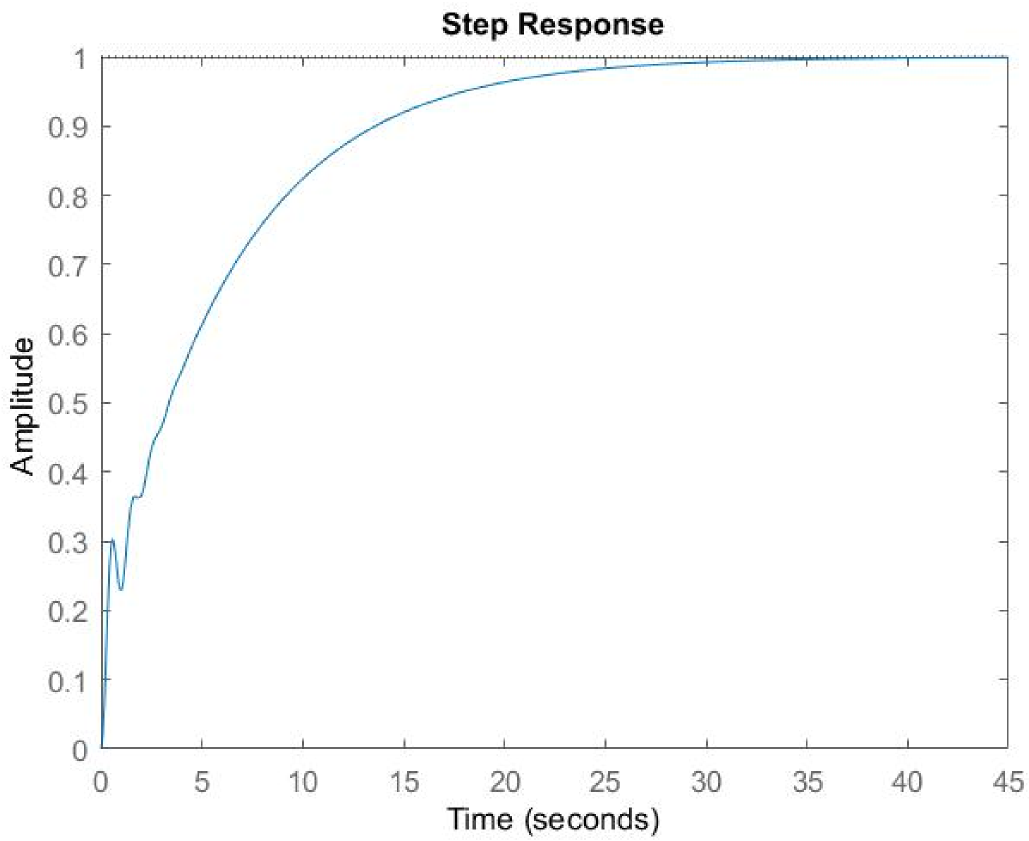


$b)$ $\left|z = \frac{1}{T_i}\right| < \left|P_{1,2}\right|$ by choosing $T_i = 1$


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
25
20
15
10
5
0
-5
-10
-15
-20
-25
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
50
0
-50
-100
0
-45
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}


Nyquist Diagram
Imaginary Axis
Real Axis
10
8
6
4
2
0
-2
-4
-6
-8
-10
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4

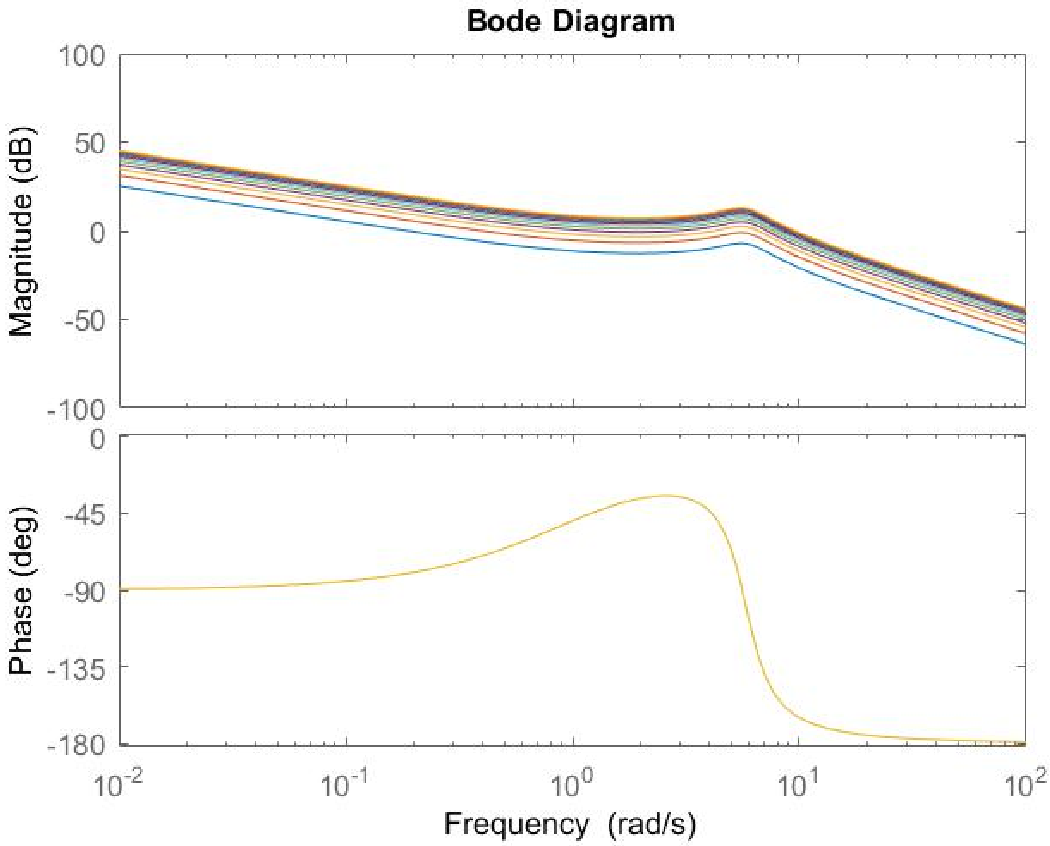

Step Response
Amplitude
Time (seconds)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
30
35
40
45

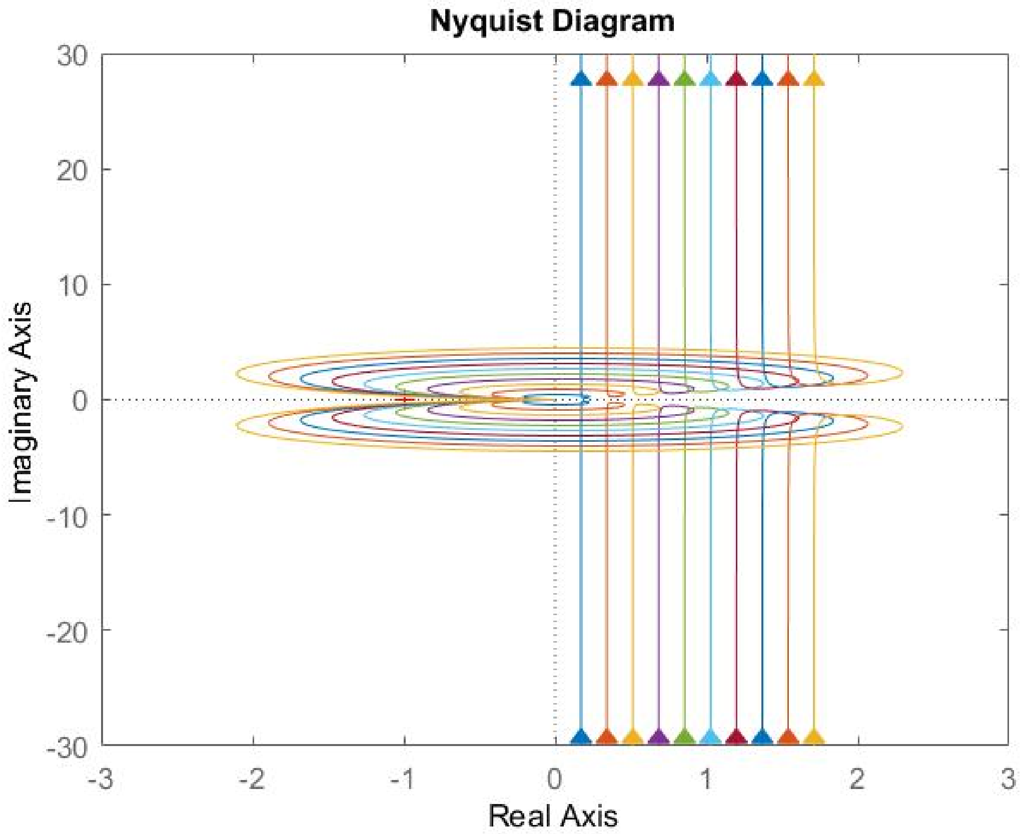
As we see the results, system is always stable for any values of k_p, and stability criterion is satisfied in these approaches (bode, nyquist, root locus).

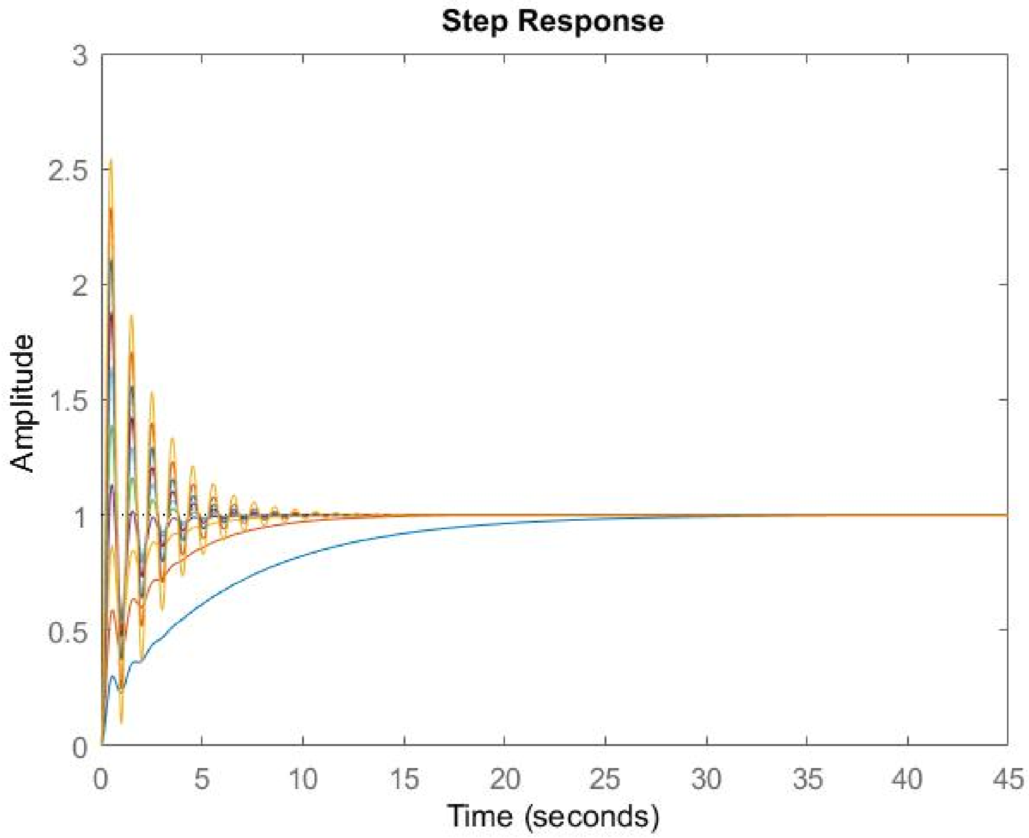
$$\begin{cases} bode: PM > 0 \ , GM \ \rightarrow +\infty \\ nyquist: P = 0 \, , N = 0 \ \rightarrow P = N \end{cases}$$

Also, from the last plot it is obvious that system can follow the step input signal by having only P controller. Therefore, steady-state error in this condition is almost zero.

<u>Effect of increasing values of gain:</u> Increasing the gain of the controller from 100 to 1000, and seeing the results.




Nyquist Diagram
30
20
10
0
-10
-20
-30
Imaginary Axis
-3
-2
-1
0
1
2
3
Real Axis


Step Response
3
2.5
2
1.5
1
0.5
0
Amplitude
0
5
10
15
20
25
30
35
40
45
Time (seconds)

Part A:

To begin with, the linearized equation of the mechanical system motion is obtained in the exercise session which is:

$$(J_m + J)\delta\dot{\omega} + (C - A)\delta\omega = B\delta y$$

Where $J^* = J_m + J$ and $C^* = C - A$

We introduce a proportional controller such as $\delta y = k_p(\delta\omega_{ref} - \delta\omega)$

System stable when $Re(\lambda) < 0$ here $\lambda = -C^*/J^*$ stable for $C^* > 0$ ie $C - A > 0$ which is true for our case where A=-0.986.

In the La place domain this becomes:

$$(J^*s + C^*)\delta\omega(s) = BK_p(\delta\omega_{ref} - \delta\omega)$$

Open loop TF:

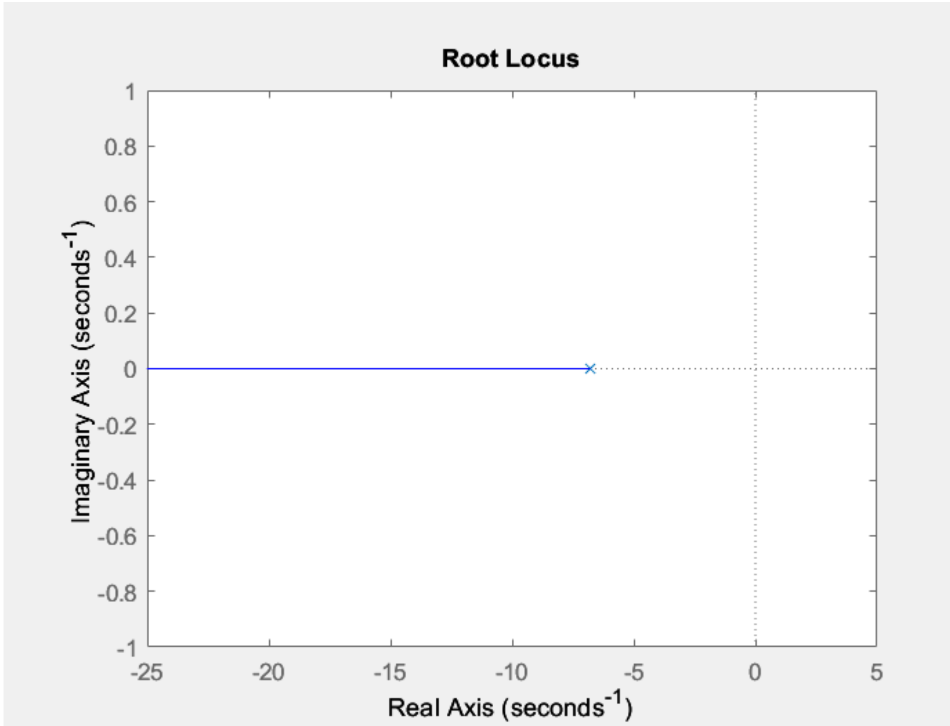
$$GH(s) = \frac{BK_p}{J^*s + C^*}$$

And Closed loop TF:

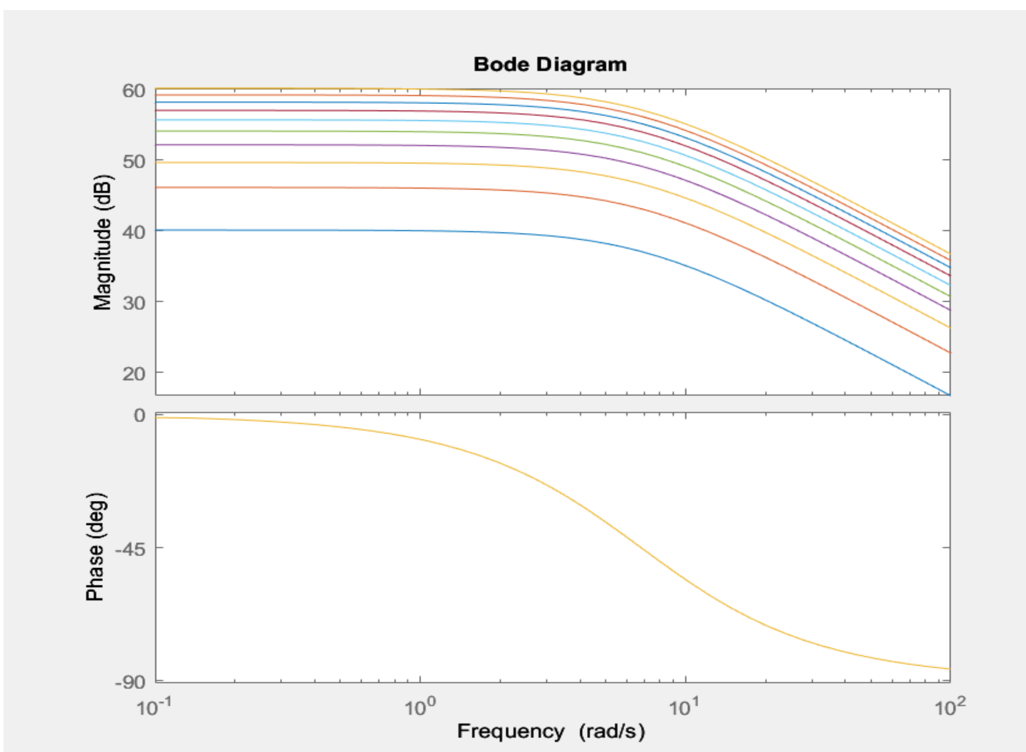
$$L(s) = \frac{G}{1+G} = \frac{BK_p}{J^*s + C^* + BK_p}$$

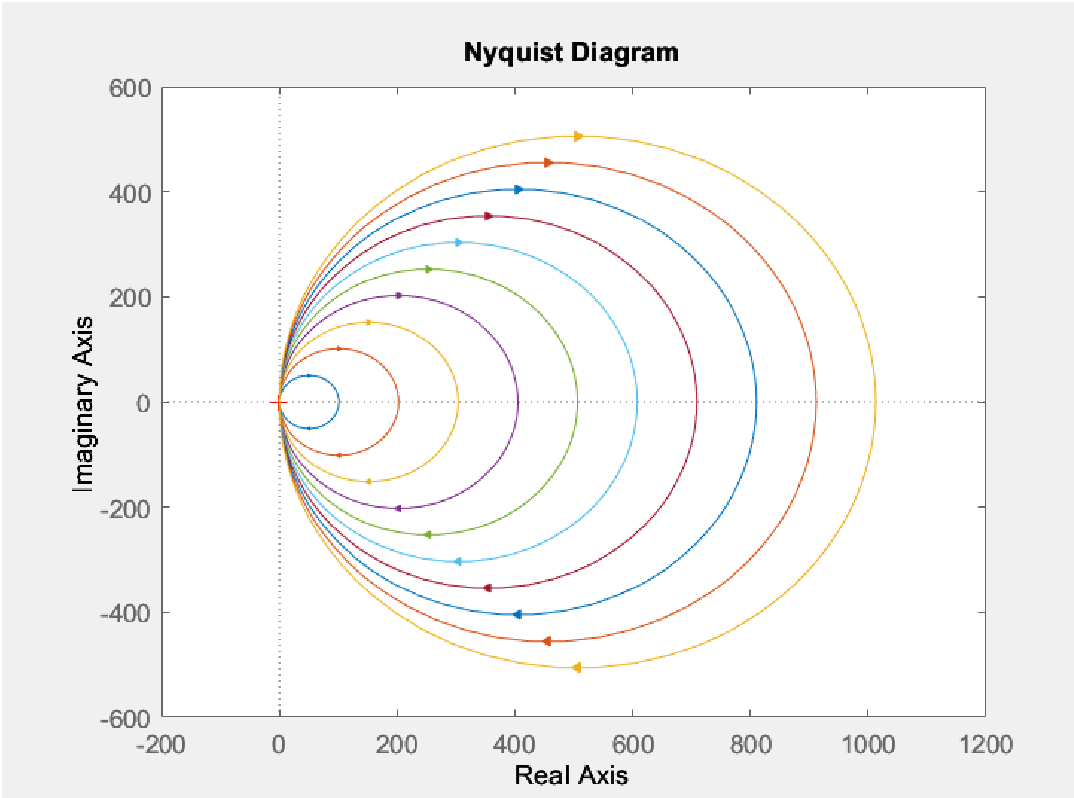
For all the following cases we start by taking kp=100, and then see the effect of increasing this value.

We will start by showing the Root locus of the system



Increasing the gain of the controller from 100 to 1000, and seeing the results



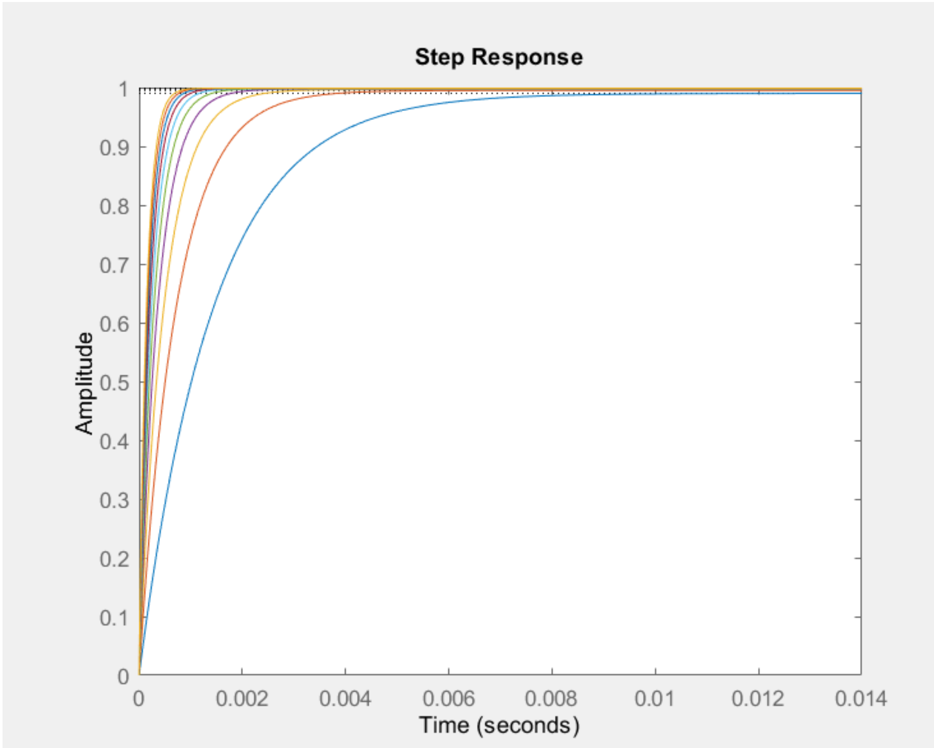


As expected, system is stable for any value of kp, and stability criterion is satisfied in all of the approaches (bode, nyquist, root locus).

$Bode: Pm \;\&\; Gm > 0$

$Nyquist: \overline{N} = P = 0$

For all values of Kp



Theoretically, we expect increase of overshoot, natural and damped frequency, and decrease of rise time and steady-state error by enhancing of k*p* value. So, we can see these changes in above plots which happens by increasing the gain. (These comments are also valid for Part B)

Part B:

In this part we have a 2 DOF system.

The equation of the system becomes:

$$\begin{bmatrix} J_m & 0 \\ 0 & J \end{bmatrix} \begin{matrix} \delta\ddot{\theta}_1 \\ \delta\ddot{\theta}_2 \end{matrix} + \begin{bmatrix} C-A & 0 \\ 0 & C \end{bmatrix} \begin{matrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \end{matrix} + \begin{bmatrix} K_T & -K_T \\ -K_T & K_T \end{bmatrix} \begin{matrix} \delta\theta_1 \\ \delta\theta_2 \end{matrix} = \begin{matrix} B \\ 0 \end{matrix} \delta y$$

Case 1: Co-located control

In this case

$$\delta y = K_p(\delta\dot{\theta}_{1ref} - \delta\dot{\theta}_1)$$

Our equation becomes:

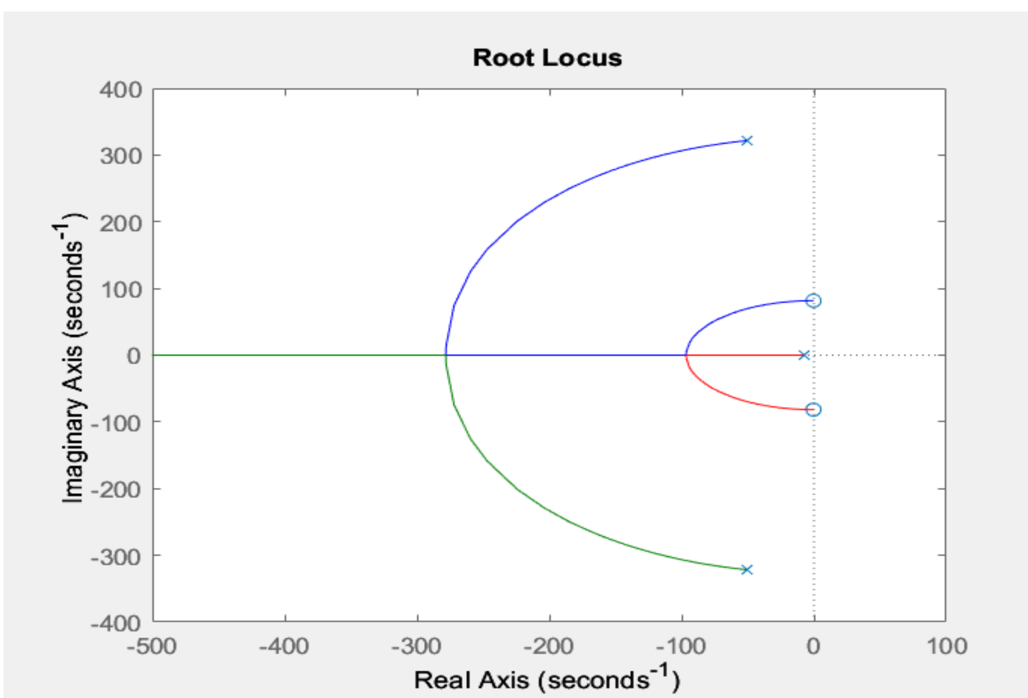
$$\begin{bmatrix} J_m & 0 \\ 0 & J \end{bmatrix} \begin{matrix} \delta\ddot{\theta}_1 \\ \delta\ddot{\theta}_2 \end{matrix} + \begin{bmatrix} C-A+BK_p & 0 \\ 0 & C \end{bmatrix} \begin{matrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \end{matrix} + \begin{bmatrix} K_T & -K_T \\ -K_T & K_T \end{bmatrix} \begin{matrix} \delta\theta_1 \\ \delta\theta_2 \end{matrix} = \begin{matrix} BK_p \\ 0 \end{matrix} \delta\dot{\theta}_{1ref}$$

We notice that by increasing Kp we can improve the stability of the system if it is already stable or stabilize the system if it is originally unstable.

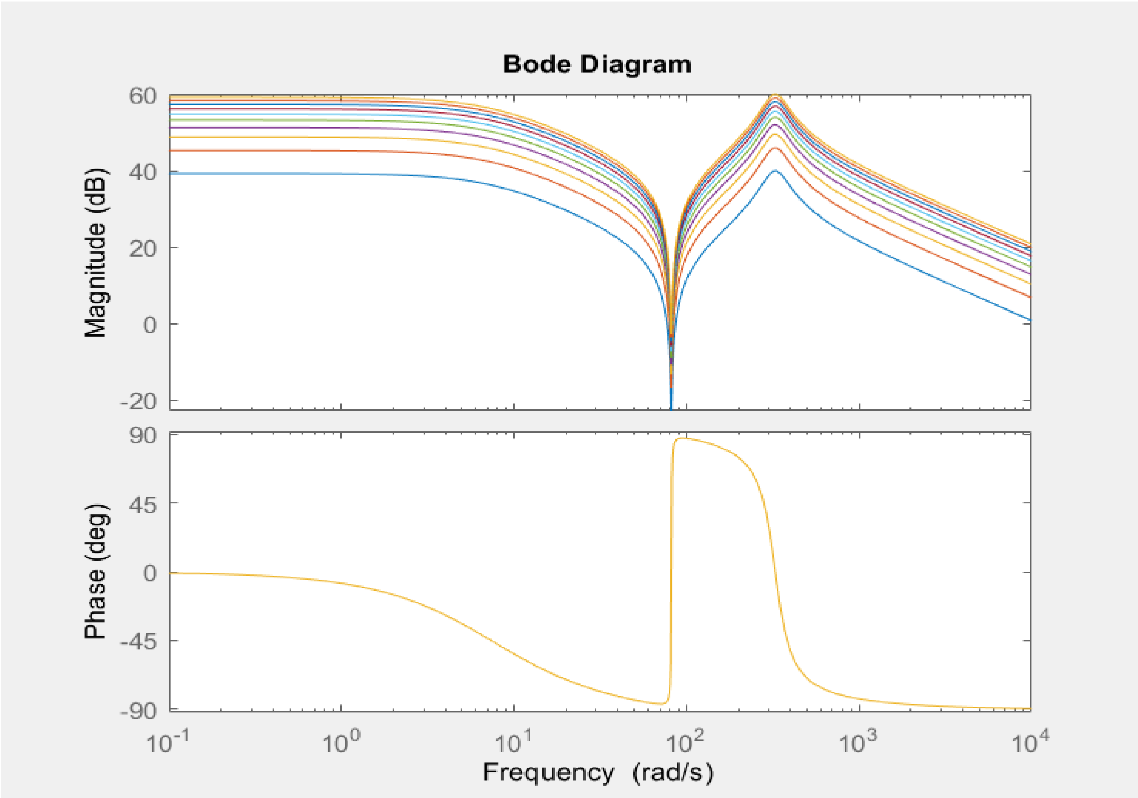
The open loop transfer function becomes (obtained in class):

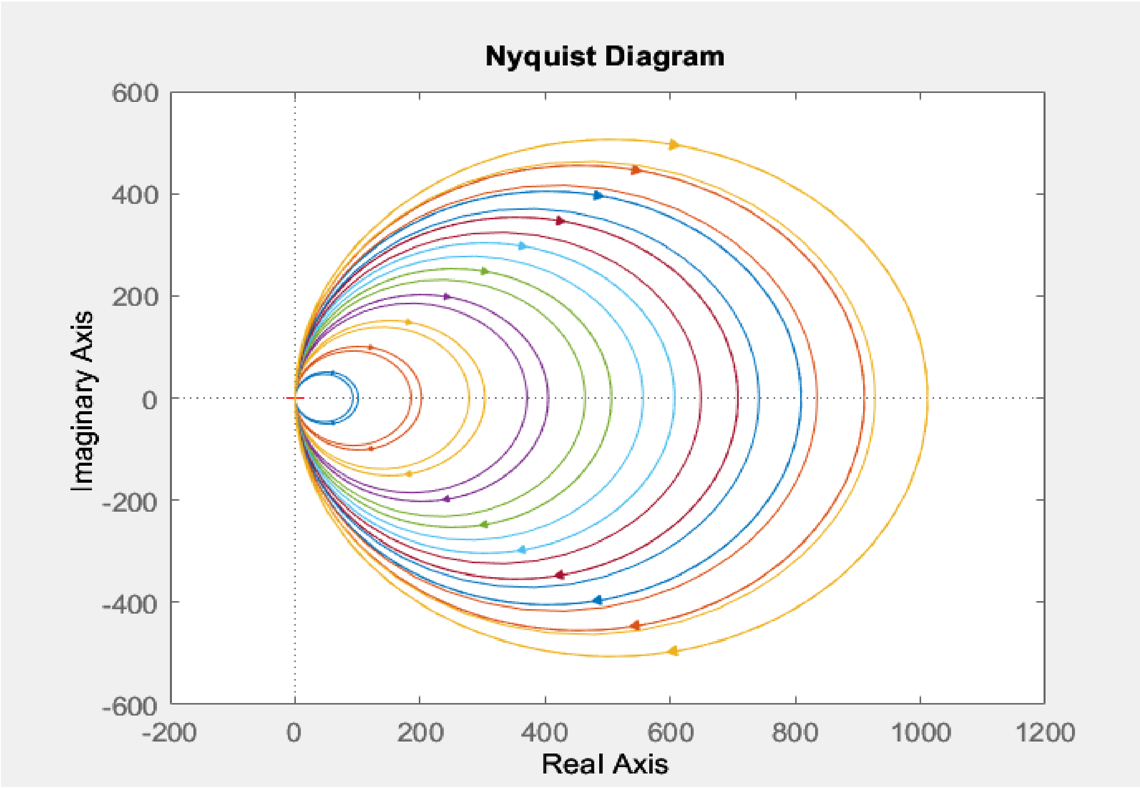
$$GH(s) = \frac{K_p B(Js^2 + Cs + K_T)}{a_4 s^3 + a_3 s^2 + a_2 s + a_1}$$

Start with the root locus:



Increasing the gain of the controller from 100 to 1000, and seeing the results:



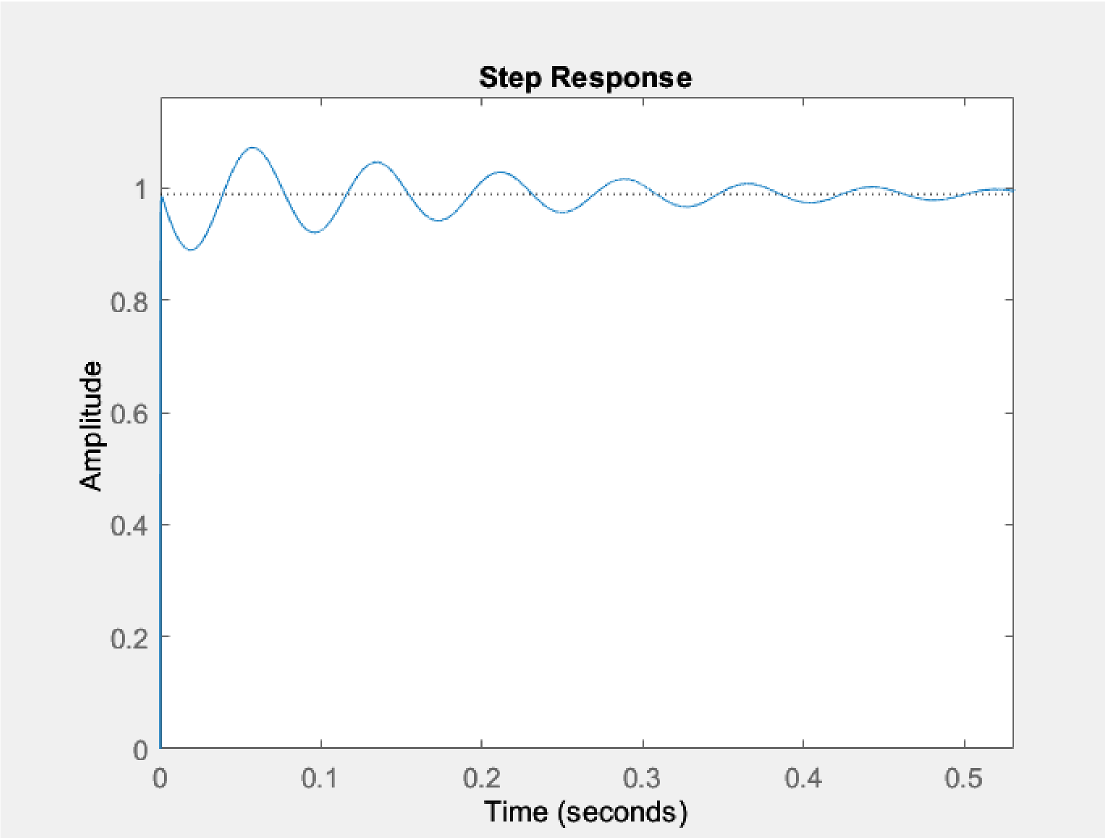


As expected, system is stable for any value of kp, and stability criterion is satisfied in all of the approaches (bode, nyquist, root locus).

$$Bode: Pm \ \& \ Gm > 0$$

$$Nyquist: \overline{N} = P = 0$$

For all values of Kp



Case 2: Non colocated control

In this case

$$\delta y = K_p(\delta\dot{\theta_{2ref}} - \delta\dot{\theta}_2)$$

Our equation becomes:

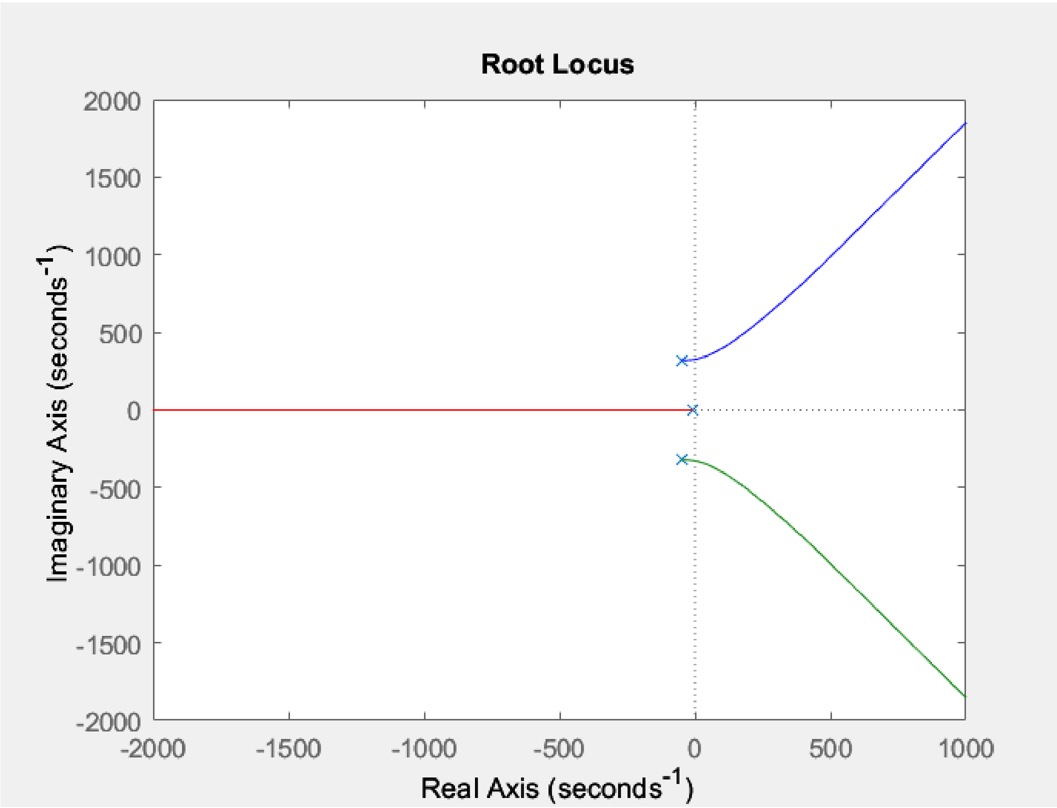
$$\begin{bmatrix} J_m & 0 \\ 0 & J \end{bmatrix} \begin{matrix} \delta\ddot{\theta}_1 \\ \delta\ddot{\theta}_2 \end{matrix} + \begin{bmatrix} C - A & BK_p \\ 0 & C \end{bmatrix} \begin{matrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \end{matrix} + \begin{bmatrix} K_T & -K_T \\ -K_T & K_T \end{bmatrix} \begin{matrix} \delta\theta_1 \\ \delta\theta_2 \end{matrix} = \begin{matrix} BK_p \\ 0 \end{matrix} \delta\dot{\theta_{1ref}}$$

Non collocated control can introduce an instability to our system (no longer a symmetric matrix).

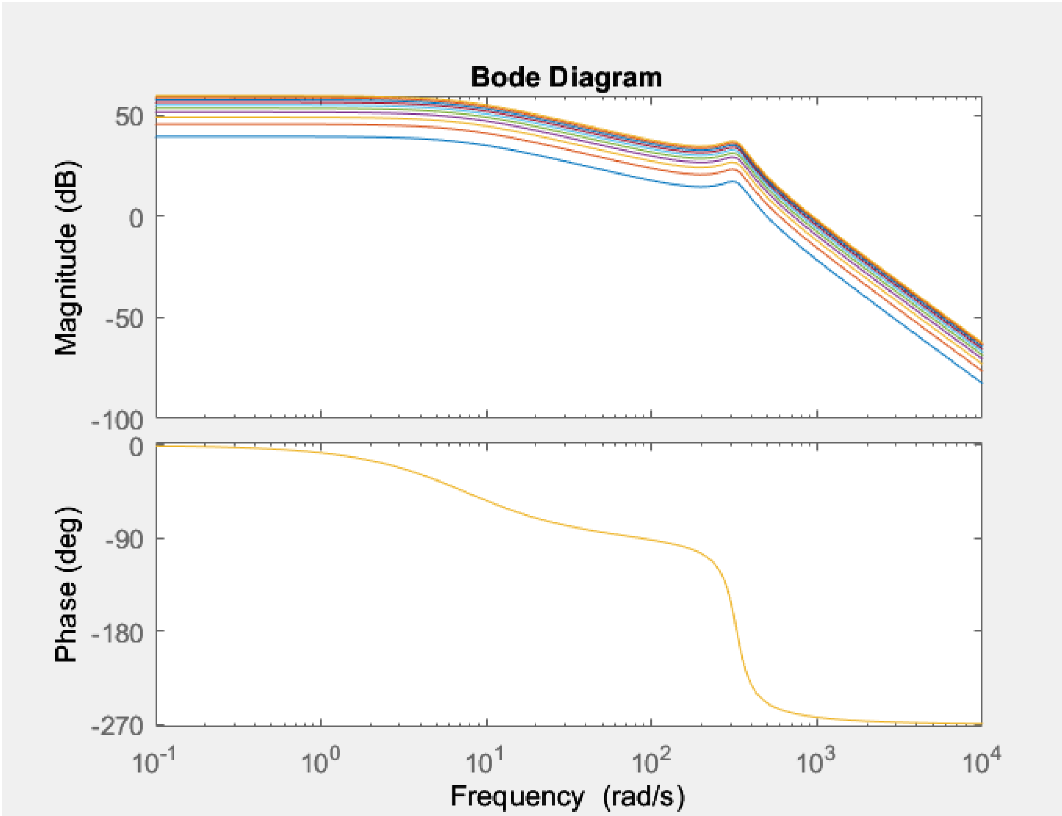
The open loop transfer function becomes (obtained in class):

$$GH(s) = \frac{K_p B K_T}{a_4 s^3 + a_3 s^2 + a_2 s + a_1}$$

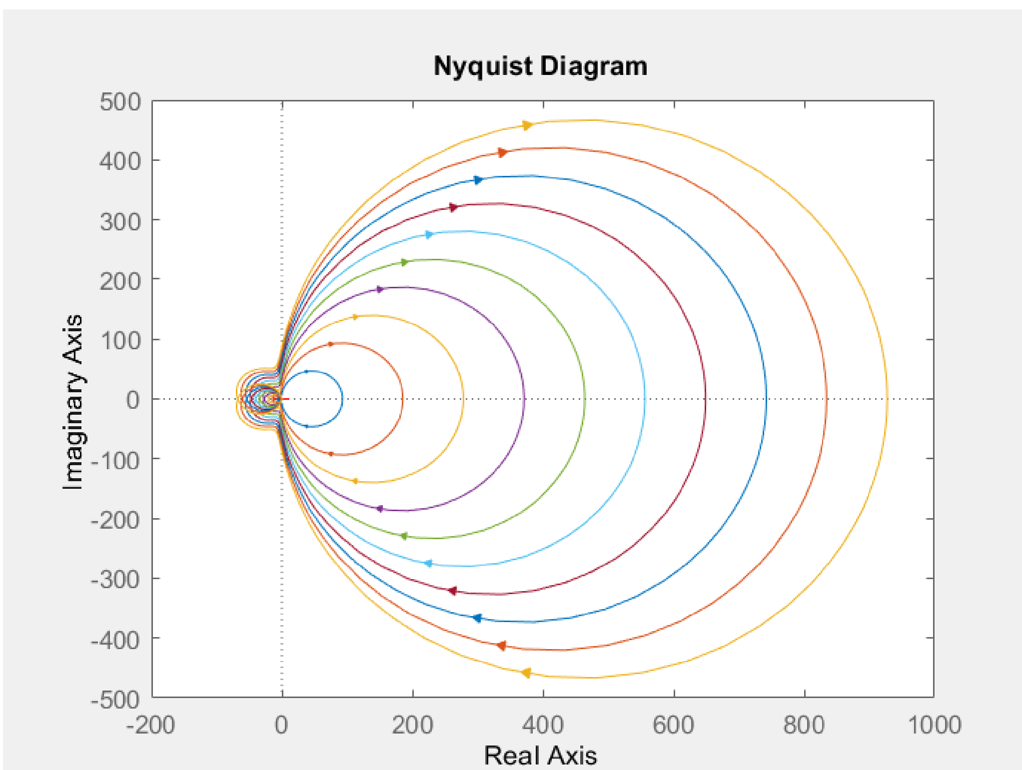
Root locus:



We can see clearly that for high values of Kp the system becomes dynamically unstable.



Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
0
-90
-180
-270
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/s)



Nyquist Diagram
Imaginary Axis
500
400
300
200
100
0
-100
-200
-300
-400
-500
-200
0
200
400
600
800
1000
Real Axis

Bode and Nyquist diagrams confirm what we saw in the root locus, that for high kp we have an instability in our system:

For a low Kp:

$$G_m and\ P_m > 0 \rightarrow Stable\ system$$

$$\overline{N} = P = 0 \rightarrow Stable$$

But for a high Kp:

$$G_m and\ P_m < 0 \rightarrow Unstable\ system$$

$$\overline{N} = 1\ and\ P = 0 \rightarrow Unstable\ system$$