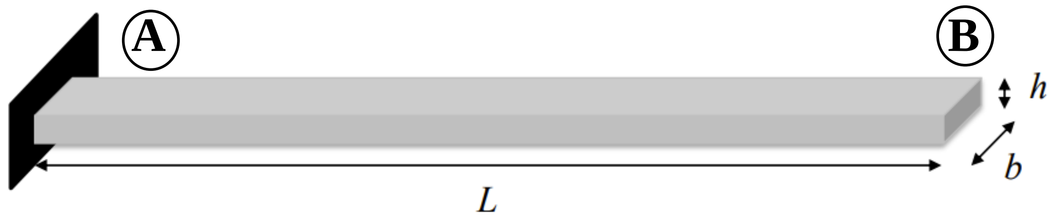
Objectives :

1. To compute the natural frequencies and the mode shapes of a cantilever beam, by referring to the standing wave solution of a slender beam in bending vibration.
2. Computing its Frequency Response Function and studying the forced response of the system.

Given Data:

Aluminum beam with rectangular cross-section



Parameter	symbol	unit	value
Lenght	L	mm	1200
Thickness	h	mm	8
Width	b	mm	40
Density	ρ	kg/m^3	2700
Young's Modulus	E	GPa	68

Except for the concentrated moments acting on the beam at the boundary, the beam undergoes no external loads during analysis. The following formula provides the standardization:

The product of Φ(x) and G(t) can be used to express the displacement of the beam at a specific point and time, w(x,t),

$$w(x,t) = \Phi(x).G(t)$$

Where,

- The mode shape, which is a function of the position x, is represented by **Φ(x)**
- The amplitude of the shape change over time is represented by **G(t).**

The theoretical equations lead to the following solution for a standing wave on a beam:

$$\mathbf{w(x,t) = [A\cos(\gamma x) + B\sin(\gamma x) + CCh(\gamma x) + DSh(\gamma x)] * \cos(\omega t + \psi)}$$

Characteristic equation

$$\gamma = \sqrt[4]{\frac{m}{EJ}}\sqrt{\omega}$$, where m= ρ.(Lbh) , J=(bh/12)

We must compute the boundary conditions to ascertain the unknown coefficients A, B, C, and D.

Boundary conditions at A:

$w(0,t) = 0$ A+C=0 **(1) – Displacement at A**

$\frac{\partial}{\partial x} w\Big|_0 = 0$ B+D=0 **(2) – Rotation at A**

Boundary conditions at B.

$M(L,t) = 0 \Rightarrow \mathrm{EJ}\frac{\partial^2}{\partial x^2} w\Big|_L = 0$ **- Bending line equation at B**

Simplified to: *-A cos γL - B sin γL + C cosh γL + D sinh γL = 0* **(3)**

$T(L,t) = 0 \Rightarrow \mathrm{EJ}\frac{\partial^3}{\partial x^3} w\Big|_L = 0$ **- Force equation at B**

Simplified to : *A sin γL - B cos γL + C sinh γL + D cosh γL=0* **(4)**

Rewriting the boundary conditions in matrix form

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos(\gamma * L) & -\sin(\gamma * L) & \cosh(\gamma * L) & \sinh(\gamma * L) \\ \sin(\gamma * L) & -\cos(\gamma * L) & \sinh(\gamma * L) & \cosh(\gamma * L) \end{bmatrix}}_{\mathbf{H(\omega)}} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \underline{0}$$

$$[H(\omega)]\ \underline{X} = \underline{0}$$

1. Computing Natural Frequencies

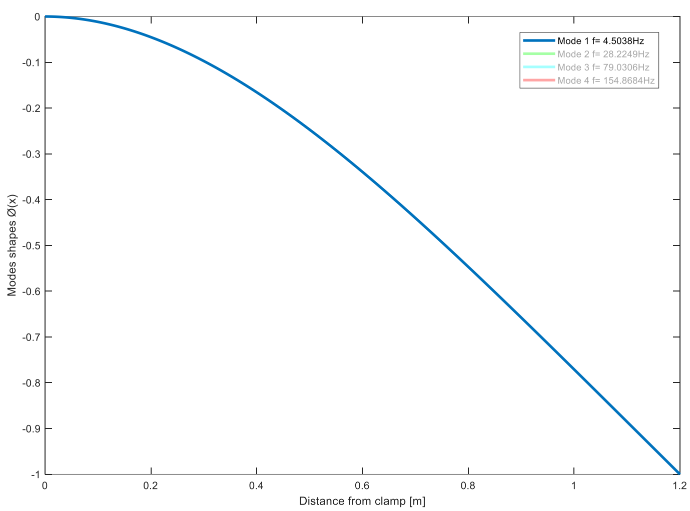
It is possible to get the solution of the system (Natural Frequencies ωi) by imposing the singularity of the coefficient matrix as det([H(ω)]) = 0 (for the solution to be non-trivial) to obtain γ and using the characteristic equation solving we obtain:-

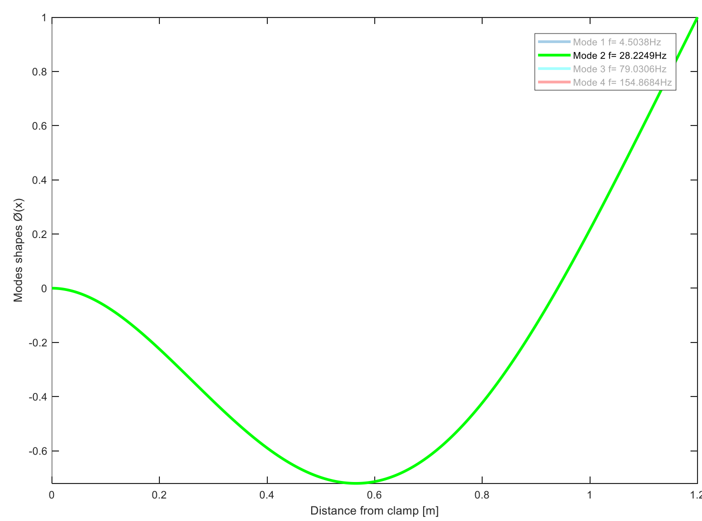
$$\omega_n = \gamma_n^2 \sqrt{\frac{EJ}{m}} \qquad \text{and} \qquad f_n = 2\pi\omega_n \qquad \text{for } n = 1, 2, ..., \infty$$

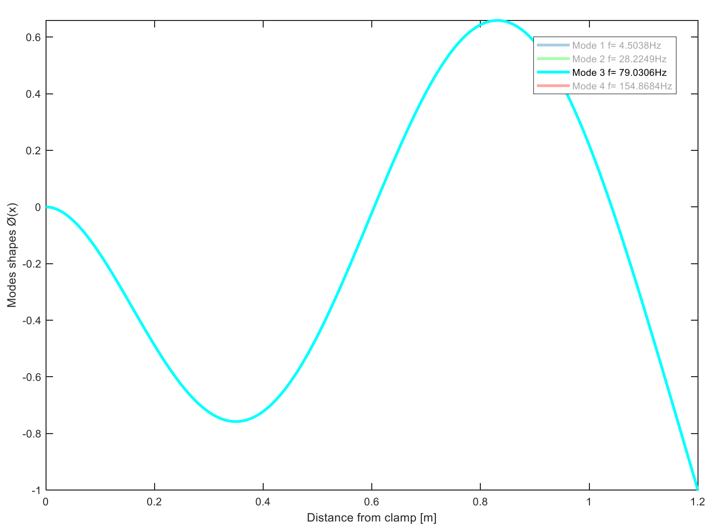
	1st	2nd	3rd	4th	5th	6th
Natural Frequencies ωi (Hz)	4.503	28.22	79.03	154.86	256	382.4

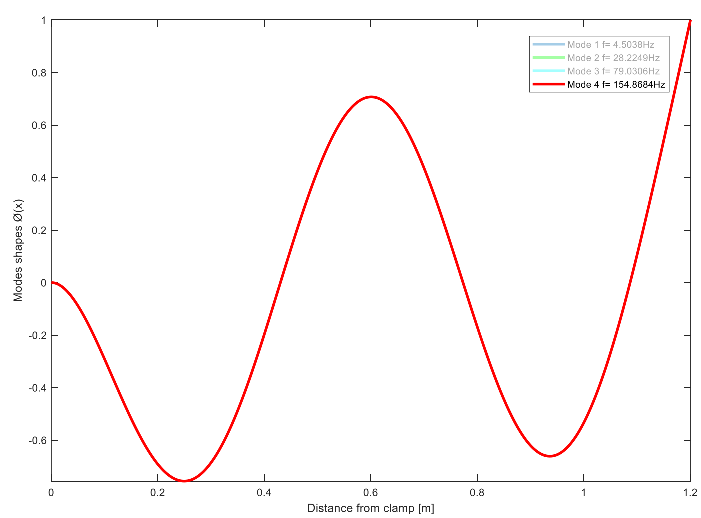
Computing Mode Shapes:

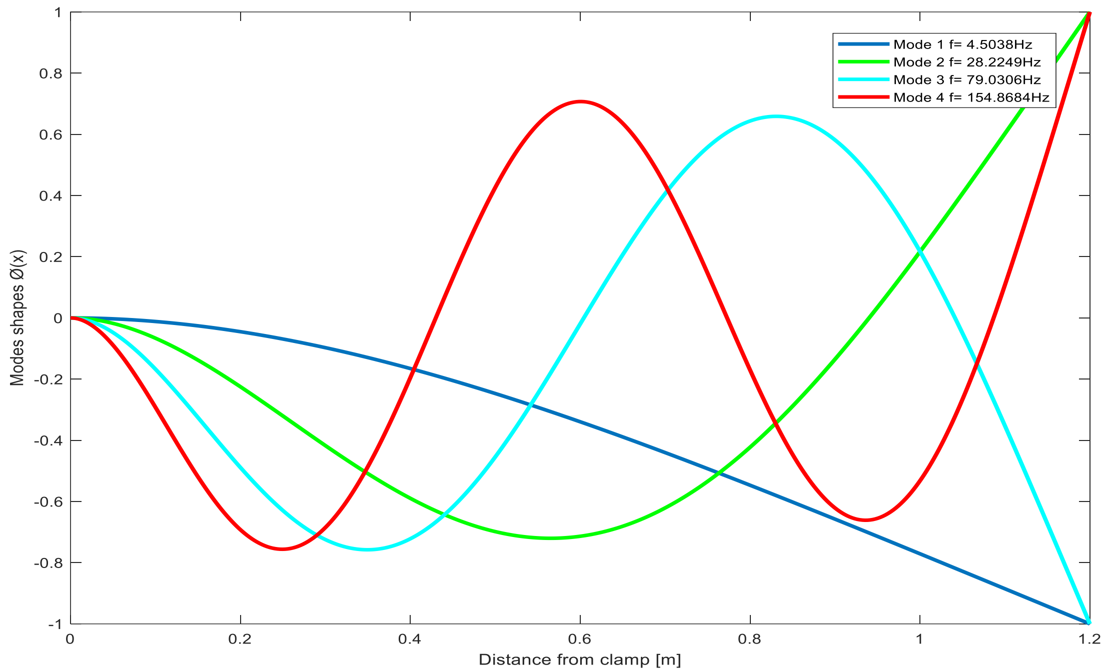
We compute the mode shapes for the first 4 natural frequencies. To solve the equations, identify the coefficients A, B, C, and D, and assuming D=1 to avoid trivial solutions, one must obtain i and replace its value in the matrix [H].



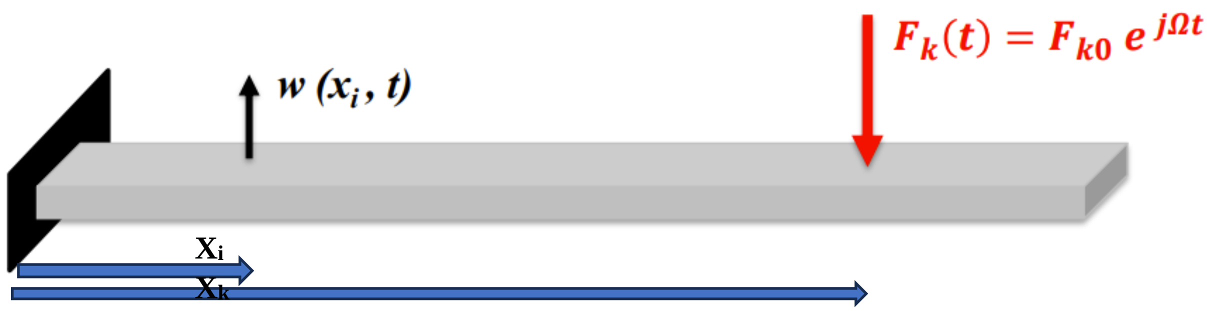








2. Frequency Response Function



Given a point load applied to the x_k cross-section of the beam, we are analyzing a forced bending vibration for the beam. To determine the system's steady-state response in terms of the physical variable w in a particular x_i cross-section of the beam is what we are interested in . The FRF is given by :

$$G(j\Omega) = \sum_{i=1}^{n} \frac{\Phi_i(x_i)\, \Phi_i(x_k)/m_i}{-\Omega^2 + j2\xi_i\omega_i\Omega + \omega_i^2}$$

Where,

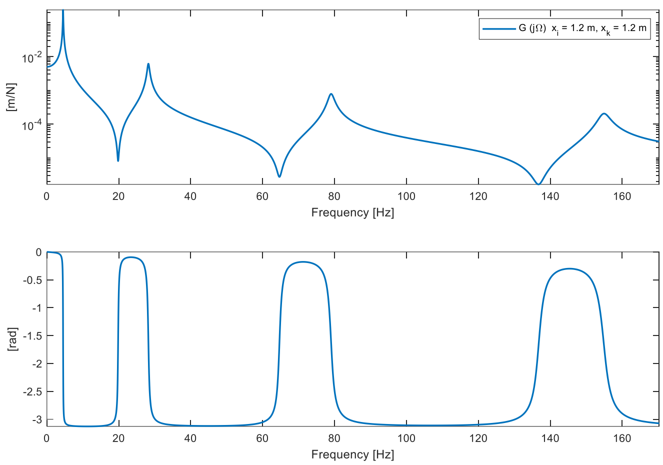
$\phi_i(x_i)$: shape functions of mode shape i at the output x_i
$\phi_i(x_k)$: shape functions of mode shape i at input positions x_k
m_i: modal mass of mode i, $m_i = \int_0^L m\, \phi_i^2(x)dx$
ω_i: natural frequency of mode i
ξ_i: damping ratio

In our simulation,

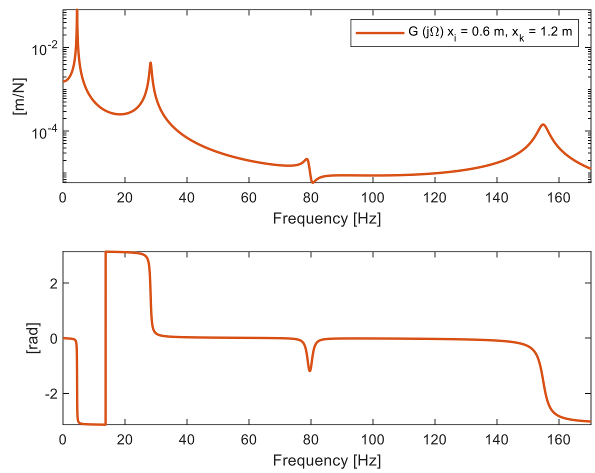
- we assume a damping ratio as **ξ = 1%**
- the following cases of input and output positions:

Input force position x_k	L/2	L/2	L	L
Observed position x_i	L/4	L	L/2	L

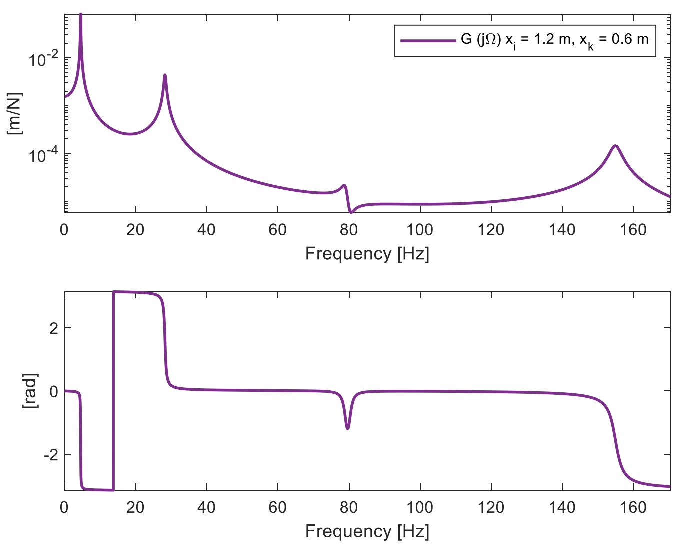
We find these FRF trends in terms of amplitude and phase concerning frequency (on the x-axis) by analyzing the system's response from various points of observation.



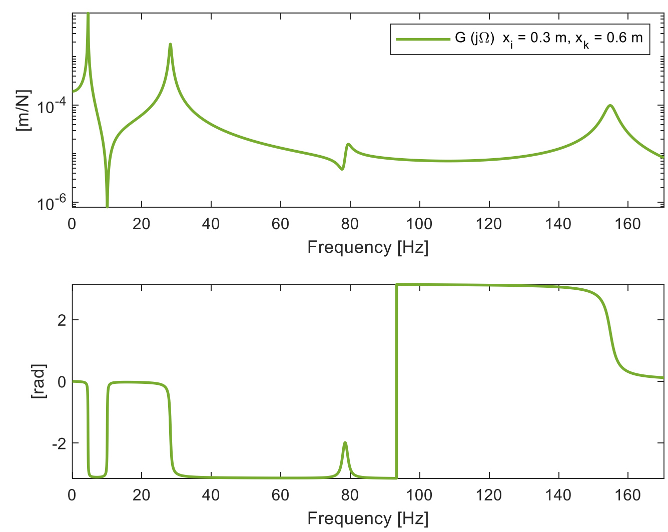
$x_i = x_k = L$



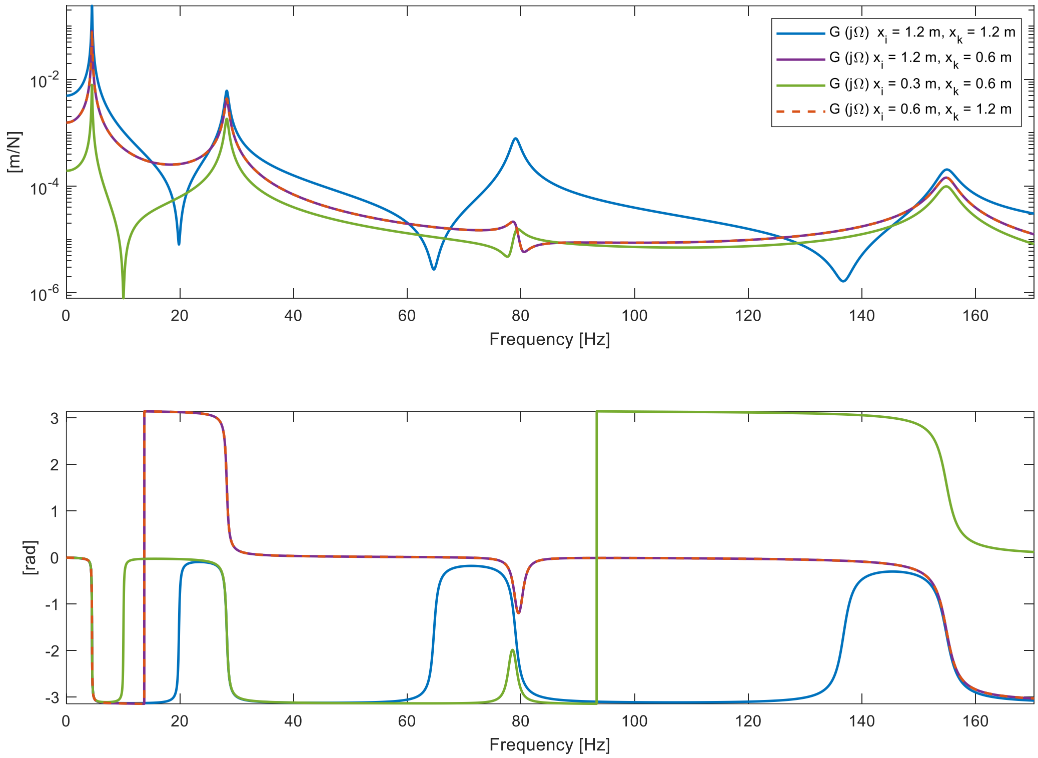
$x_i = L/2,\ x_k = L$



$x_i = L,\ x_k = L/2$



$x_i = L/4,\ x_k = L/2$



When taking a colocated FRF into account, we compute the response in the same spot where the force was applied. The peaks of the FRF for colocated points i.e., $x_i = x_k = L$ (Blue Line) we can identify the four natural frequencies. In the corresponding phase diagram, we can observe a phase difference of $-\pi$. The valleys represent frequencies wherein the system does not exhibit any significant response to the input force.

The two FRFs coincide when plotted along the lines of $x_i = L/2$, $x_k = L$ (Orange Line) and $x_i = L$, $x_k = L/2$ (Purple line). Analysis of the FRF function $G(j\Omega)$ enables one to reason in this way. The $\Phi_i(x_i)\Phi_i(x_k)$ is identical in both situations. This mode's peak cannot be found at the third natural frequency, which is 79 Hz. When the input and/or output are placed in a position that corresponds to a node of a vibration mode, neither the peak nor a decrease in the phase diagram occurs.

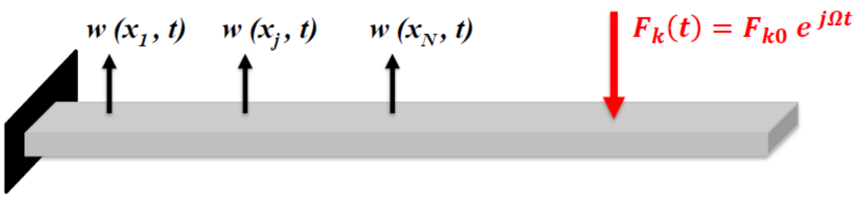
Finally, the case wherein $x_i = L/4$ and $x_k = L/2$ (Green line) behaves similarly to the previous case. In fact, not only can it be seen that there is no peak for ω_3, but also that the peak that corresponds to ω_2 has a smaller value compared to the previous peaks. This is due to the proximity of the input force and output measurement to the respective nodes of these vibration modes.

Part 1B – Experimental Modal Analysis

Objective:

1. Assuming the FRFs computed in Part 1A to be representative of an experimental test carried out on a cantilever beam, apply modal parameters identification according to the single mode procedure.

- Natural frequency
- Damping ratio
- Mode shape



a. Natural frequency calculation :

With the magnitude of FRF, find maxima, identified from a position of a resonance peak in the specified frequency range. The corresponding frequency of the maxima is the natural frequency of the system.

b. Damping ratio calculation:

1. Half-Power Point

We first recognize the maximum in the magnitude diagram and the corresponding frequency ω_i

Then we identify the two frequencies ω_1 and ω_2 at which the magnitude of the FRF drops down by a factor of $\frac{1}{\sqrt{2}}$

$$\xi_i = \frac{\omega_2^2 - \omega_1^2}{4\omega_i^2}$$

2. The slope of the phase diagram

The damping ratio can be identified from the slope of the phase diagram, in correspondence with the natural frequency related to the mode shape

For each slope and ω_0 (corresponding to each mode), we have to use the following equation to calculate the corresponding Damping Ratio ξ.

$$\xi_i = -\frac{1}{\omega_i \frac{\partial \phi}{\partial \Omega}|_{\Omega=\omega_i}}$$

c. **Mode shape**

Using the calculated parameters, natural frequencies and damping ratio, to identify the mode shapes. The mode shapes are identified from the imaginary part of experimental FRFs and evaluated for variable output node using:

$$G(\Omega) = \sum_{i=1}^{n} \frac{X_j^{(i)} X_k^{(i)}/m_i}{-\Omega^2 + j\Omega 2\omega_i \xi_i + \omega_i^2}$$

At resonance $\Omega = \omega i$ and approximating,

$$G \approx -j\left(\frac{X_k^{(i)}/m_i}{2\xi_i \omega_i^2}\right) X_j^{(i)}$$

Experimental results:

The values calculated using these two different approaches will almost always be slightly out of sync.

To identify different values for the half-power points method and slope of the phase diagram, we impose a 1% damping ratio.

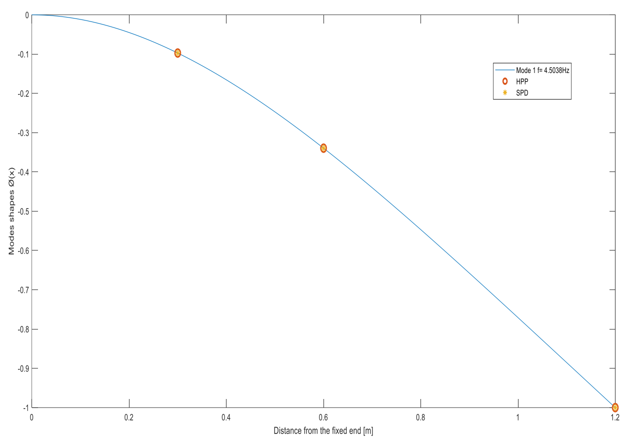
In our representations, we compute four frequencies and four force positions to various observed points.

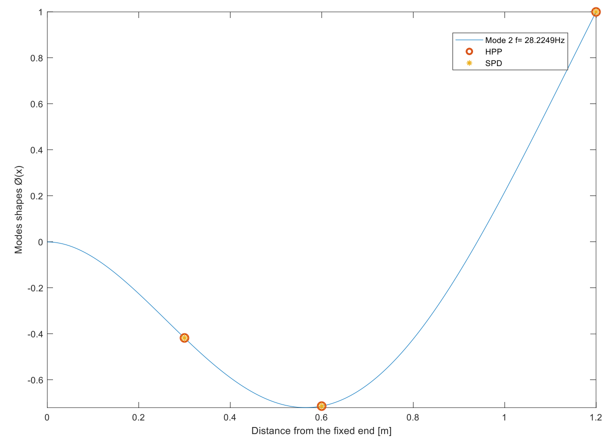
Natural Frequencies comparison :

	Mode 1	Mode 2	Mode 3	Mode 4
Model	4.5038	28.2249	79.030	154.868
Numerical Experiment	4.4985	28.2163	79.0303	154.8398

Damping Ratio (%)

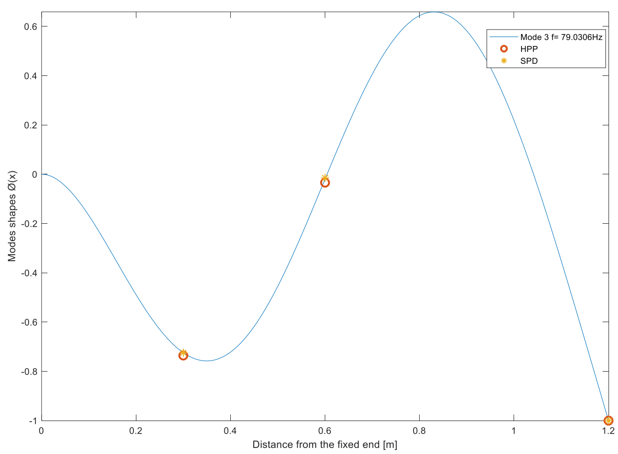
	Mode 1	Mode 2	Mode 3	Mode 4
	0.01	0.01	0.01	0.01
Half-Power Point	0.01013	0.01001	0.0100	0.0100
The slope of the phase diagram	0.0101	0.0100	0.0181	0.0100

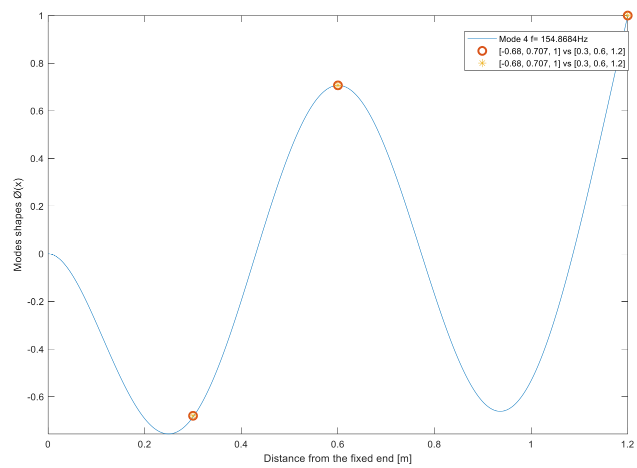




Mode shape 1

Mode shape 2





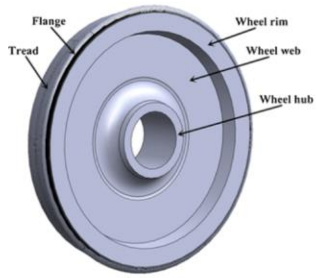
Mode shape 3

Mode shape 4

Both approaches allow us to properly compute the mode shapes to a level of accuracy. Slight deviations due to the resolution of Ω.

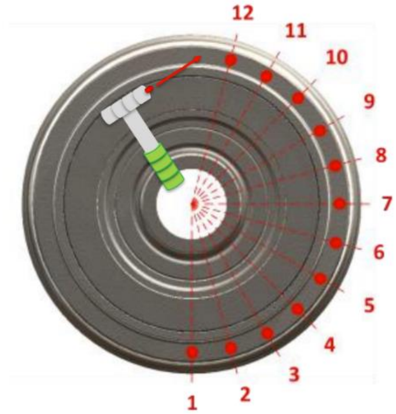
These graphs can also be plotted for the mode shape -$\Phi(x)$ for each natural frequency doing so proves that one of the constants in the function is arbitrary. In addition to this, the $\Phi(x)$ function of the standing wave solution represents the shape that the beam assumes by exciting the structure with a certain frequency at every time t but the amplitude of such shape is a function of the time.

2. Rail vehicle resilient wheel



Resilient wheel

Wheelset



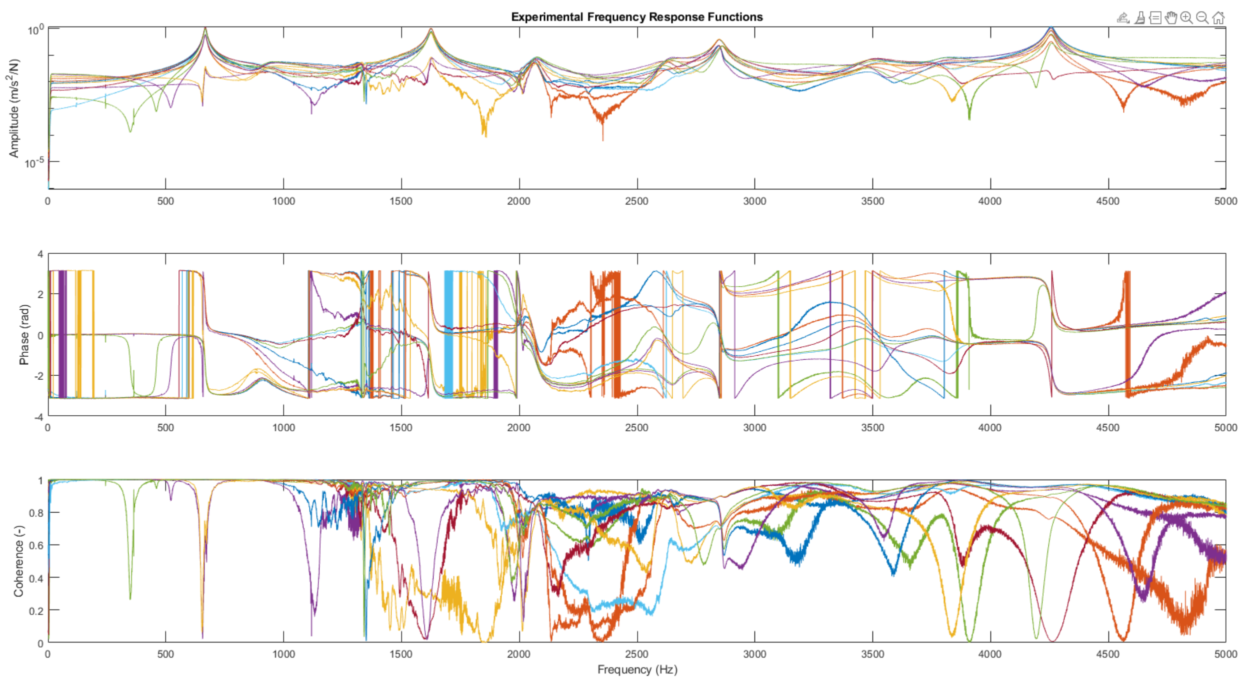
- Resilient wheel suspended through elastic support (free-free system)
- The experiment was conducted by striking the beam with an impact hammer in line with precise points.
- Due to the symmetry of the structure,12 measurement positions have been considered, which are located only on half of the wheel, with a regular angular spacing of 15°, while the vibration was recorded using piezoelectric.
- In correspondence to our experimental setup, we can therefore compute 48 different damping ratios. Similar to before, to find natural frequencies, we first determined the local maximum values of FRF. Nearly unaffected by the excitation position are these natural frequencies. As a result, each mode corresponds to 4 different natural frequencies.

With the use of the given experimental data observations from these accelerometers (**File data. mat),** the natural frequencies, coherence function and the resulting FRFs are plotted.

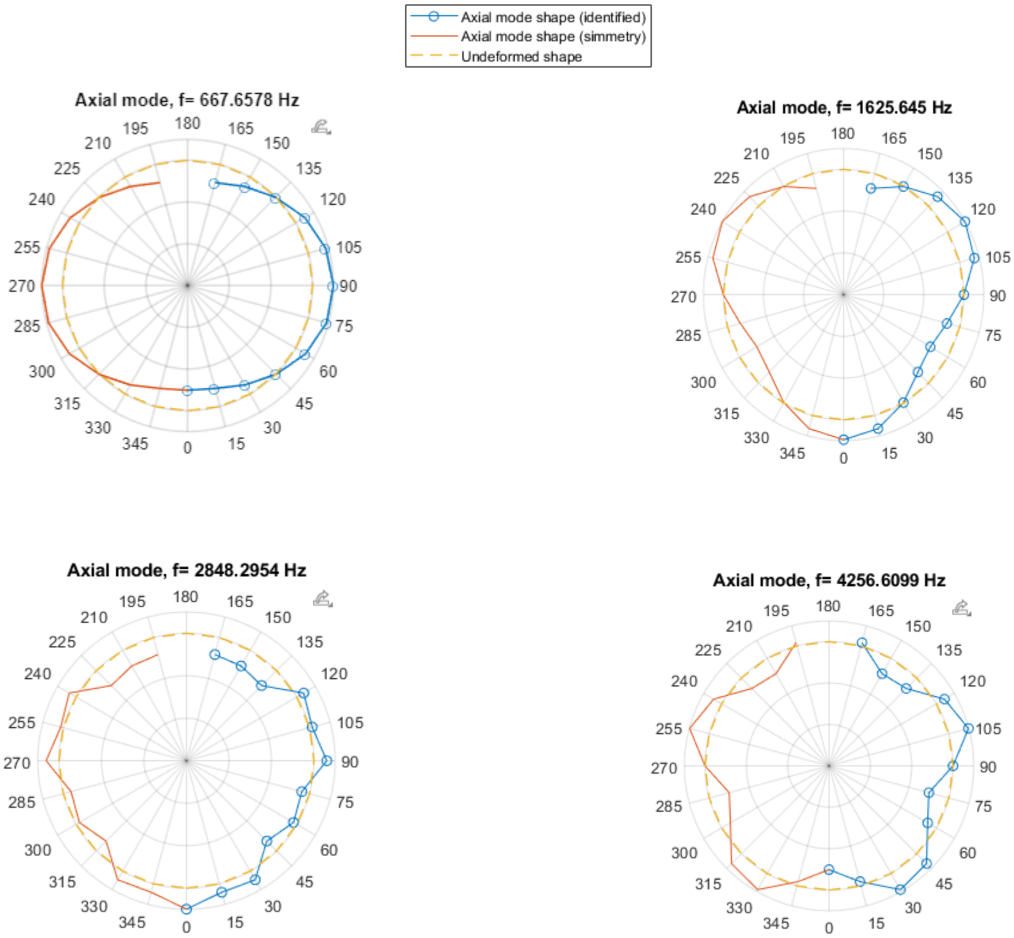
	Natural Frequencies [Hz]			
Channel	ω_1	ω_2	ω_3	ω_4
1	667.6577	1625.645	2848.295	4256.601
2	667.6577	1625.645	2846.295	4256.601
3	667.6577	1625.645	2860.295	4256.601
4	667.6577	1625.645	2848.295	4256.601
5	667.6577	1625.645	2848.295	4256.601
6	667.6577	1625.645	2860.295	4256.601
7	667.6577	1625.645	2848.295	4233.610
8	667.6577	1625.645	2848.295	4256.601
9	667.6577	1625.645	2860.628	4256.601
10	667.6577	1625.645	2848.295	4256.601
11	667.6577	1625.645	2845.295	4256.601
12	667.6577	1625.645	2862.295	4260.601

	Damping Ratio (%)			
Channel	$\xi_{\omega 1}$	$\xi_{\omega 2}$	$\xi_{\omega 3}$	$\xi_{\omega 4}$
1	0.80	0.61	0.64	0.36
2	0.80	0.62	0.45	0.37
3	0.81	0.55	0.62	0.35
4	0.66	0.60	0.65	0.36
5	0.79	0.61	0.44	0.33
6	0.80	0.61	0.61	0.36
7	0.80	0.65	0.61	-
8	0.80	0.60	0.46	0.36
9	0.80	0.61	0.61	0.37
10	0.65	0.60	0.63	0.35
11	0.80	0.71	0.45	0.35
12	0.80	0.62	0.66	0.37

Concerning axial modes, we can identify four very sharp resonance peaks, the first of which has a frequency of around 700 Hz, the second around 1600 Hz, the third around 2800 Hz, and the fourth at roughly 4300 Hz.



We plot the modes of vibration and use the symmetry to replicate the shape for the other half of the wheel.



Axial mode shape (identified)
Axial mode shape (simmetry)
Undeformed shape
Axial mode, f= 667.6578 Hz
Axial mode, f= 1625.645 Hz
Axial mode, f= 2848.2954 Hz
Axial mode, f= 4256.6099 Hz
0 15 30 45 60 75 90 105 120 135 150 165 180 195 210 225 240 255 270 285 300 315 330 345